

EXISTENCE OF SOLUTIONS OF SCALAR FIELD EQUATIONS WITH FRACTIONAL OPERATOR

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ABSTRACT. In this paper, the existence of least energy solution and infinitely many solutions is proved for the equation $(1 - \Delta)^\alpha u = f(u)$ in \mathbf{R}^N where $0 < \alpha < 1$, $N \geq 2$ and $f(s)$ is a Berestycki–Lions type nonlinearity. The characterization of the least energy by the mountain pass value is also considered and the existence of optimal path is shown. Finally, exploiting these results, the existence of positive solution for the equation $(1 - \Delta)^\alpha u = f(x, u)$ in \mathbf{R}^N is established under suitable conditions on $f(x, s)$.

1. INTRODUCTION

In this paper, we are concerned with the existence of nontrivial solutions of

$$(1) \quad \begin{cases} (1 - \Delta)^\alpha u = f(x, u) & \text{in } \mathbf{R}^N, \\ u \in H^\alpha(\mathbf{R}^N) \end{cases}$$

where $N \geq 2$ and $0 < \alpha < 1$. The fractional operator $(1 - \Delta)^\alpha u$ is defined by

$$(1 - \Delta)^\alpha u := \mathcal{F}^{-1} \left((1 + 4\pi^2 |\xi|^2)^\alpha \hat{u}(\xi) \right), \quad \hat{u}(\xi) := (\mathcal{F}u)(\xi) = \int_{\mathbf{R}^N} e^{-2\pi i x \cdot \xi} u(x) dx$$

and $H^\alpha(\mathbf{R}^N)$ a fractional Sobolev space consisted by real valued functions, that is,

$$(2) \quad H^\alpha(\mathbf{R}^N) := \left\{ u \in L^2(\mathbf{R}^N, \mathbf{R}) \mid \|u\|_\alpha^2 := \int_{\mathbf{R}^N} (4\pi^2 |\xi|^2 + 1)^\alpha |\hat{u}|^2 d\xi < \infty \right\}.$$

Throughout this paper, we deal with a weak solution of (1), namely, a function $u \in H^\alpha(\mathbf{R}^N)$ satisfying

$$(3) \quad \int_{\mathbf{R}^N} (4\pi^2 |\xi|^2 + 1)^\alpha \hat{u}(\xi) \overline{\hat{\varphi}(\xi)} d\xi - \int_{\mathbf{R}^N} f(x, u(x)) \varphi(x) dx = 0 \quad \text{for all } \varphi \in H^\alpha(\mathbf{R}^N)$$

where \bar{a} denotes the complex conjugate of a .

The operator $(1 - \Delta)^\alpha$ is related to the pseudo-relativistic Schrödinger operator $(m^2 - \Delta)^{1/2} - m$ ($m > 0$) and recently a lot of attentions are paid for equations involving them. Here we refer to [2–4, 12–17, 21, 23, 32, 34, 38] and references therein for more details and physical context of $(1 - \Delta)^\alpha$. In these papers, the authors study the existence of nontrivial solution and infinitely many solutions for the equations with $(m^2 - \Delta)^\alpha$ and various nonlinearities.

This paper is especially motivated by two papers [23] and [34]. In [23], the existence of positive solution of (1) is proved under the following conditions on $f(x, s)$:

- (i) $f \in C(\mathbf{R}^N \times \mathbf{R}, \mathbf{R})$.
- (ii) For all $x \in \mathbf{R}^N$, $f(x, s) \geq 0$ if $s \geq 0$ and $f(x, s) = 0$ if $s \leq 0$.
- (iii) The function $s \mapsto s^{-1} f(x, s)$ is increasing in $(0, \infty)$ for all $x \in \mathbf{R}^N$.
- (iv) There are $1 < p < 2_\alpha^* - 1 = (N + 2\alpha)/(N - 2\alpha)$ and $C > 0$ such that $|f(x, s)| \leq C|s|^p$ for all $(x, s) \in \mathbf{R}^N \times \mathbf{R}$.
- (v) There exists a $\mu > 2$ such that $0 < \mu F(x, s) \leq s f(x, s)$ for all $(x, s) \in \mathbf{R}^N \times (0, \infty)$ where $F(x, s) := \int_0^s f(x, t) dt$.
- (vi) There exist continuous functions $\bar{f}(s)$ and $a(x)$ such that \bar{f} satisfies (i)–(v) and $0 \leq f(x, s) - \bar{f}(s) \leq a(x)(|s| + |s|^p)$ for all $(x, s) \in \mathbf{R}^N \times \mathbf{R}$, $a(x) \rightarrow 0$ as $|x| \rightarrow \infty$ and $\mathcal{L}^N(\{x \in \mathbf{R}^N \mid f(x, s) > \bar{f}(s) \text{ for all } s > 0\}) > 0$ where \mathcal{L}^N denotes the n -dimensional Lebesgue measure.

On the other hand, in [34], the author obtains a nontrivial solution of (1) with $f(x, s) = \lambda b(x)|u|^{p-1}u + c(x)|u|^{q-1}u$ under different conditions on $b(x), c(x), p, q$ where $\lambda > 0$ is a constant. Among other things, under $1 < p, q < 2_\alpha^* - 1$ and some strict inequality for the mountain pass value (the infimum of the functional on the

2010 *Mathematics Subject Classification.* 35J60, 35S05.

Key words and phrases. variational method, mountain pass theorem, symmetric mountain pass theorem, the Pohozaev identity.

Nehari manifold), the existence of positive solution of (1) is shown. However, there is no specific information when the strict inequality holds.

Our aim of this paper is to observe whether we can handle the more general nonlinearity $f(x, s)$ in (1) and obtain a positive solution. First, we treat the case where $f(x, s) = f(s)$ is a Berestycki–Lions type nonlinearity, that is, $f(s)$ satisfying (f1)–(f4) below. These conditions are introduced in [7, 8] (cf. [6]) for the case $\alpha = 1$ and almost optimal for the existence of nontrivial solution. We shall prove the existence of infinitely many solutions and least energy solutions with the Pohozaev identity, that the least energy coincides with the mountain pass value and that there is an optimal path. These properties are shown in [24, 26] for the case $\alpha = 1$. Second, we deal with the case $f(x, s)$ depends on x . Here, exploiting the optimal path and characterization by the mountain pass value, we show the existence of positive solution of (1), which generalizes the result in [23] and enables us to find a simpler sufficient condition than that of [34] in some case. See the comments after Remark 1.4.

As stated in the above, we first consider the case $f(x, s) \equiv f(s)$ and (1) becomes

$$(4) \quad \begin{cases} (1 - \Delta)^\alpha u = f(u) & \text{in } \mathbf{R}^N, \\ u \in H^\alpha(\mathbf{R}^N). \end{cases}$$

For (4), we assume that the nonlinearity f is a Berestycki–Lions type ([7, 8]):

(f1) $f \in C(\mathbf{R}, \mathbf{R})$ and $f(s)$ is odd.

(f2) $-\infty < \liminf_{s \rightarrow 0} \frac{f(s)}{s} \leq \limsup_{s \rightarrow 0} \frac{f(s)}{s} < 1$.

(f3)

$$\lim_{|s| \rightarrow \infty} \frac{|f(s)|}{|s|^{2_\alpha^* - 1}} = 0 \quad \text{where } 2_\alpha^* := \frac{2N}{N - 2\alpha}.$$

(f4) There exists an $s_0 > 0$ such that

$$F(s_0) - \frac{1}{2}s_0^2 > 0 \quad \text{where } F(s) := \int_0^s f(t)dt.$$

Notice that under (f1)–(f3), (4) has variational structure, namely, a solution of (4) is characterized as a critical point of the following functional (see Lemma 2.1)

$$(5) \quad I(u) := \frac{1}{2}\|u\|_\alpha^2 - \int_{\mathbf{R}^N} F(u)dx \in C^1(H^\alpha(\mathbf{R}^N), \mathbf{R}).$$

Our first result is the existence of infinitely many solutions of (4) and the characterization of least energy solutions by the mountain pass structure.

Theorem 1.1. *Assume $N \geq 2$, $0 < \alpha < 1$ and (f1)–(f4).*

(i) *There exist infinitely many solutions $(u_n)_{n=1}^\infty$ of (4) satisfying $I(u_n) \rightarrow \infty$ and the Pohozaev identity $P(u_n) = 0$ where*

$$(6) \quad P(u) := \frac{N - 2\alpha}{2} \int_{\mathbf{R}^N} (1 + 4\pi^2|\xi|^2)^\alpha |\widehat{u}|^2 d\xi - N \int_{\mathbf{R}^N} F(u)dx + \alpha \int_{\mathbf{R}^N} (1 + 4\pi^2|\xi|^2)^{\alpha-1} |\widehat{u}|^2 d\xi.$$

Moreover, $u_1(x) > 0$ for all $x \in \mathbf{R}^N$.

(ii) *Assume either $\alpha > 1/2$ or $f(s)$ is locally Lipschitz continuous. Then every solution of (4) satisfies the Pohozaev identity $P(u) = 0$.*

(iii) *For the following quantities*

$$\begin{aligned} c_{\text{MP}} &:= \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I(\gamma(t)), \quad \Gamma := \{\gamma \in C([0, 1], H^\alpha(\mathbf{R}^N)) \mid \gamma(0) = 0, I(\gamma(1)) < 0\}, \\ c_{\text{LES}} &:= \inf \{I(u) \mid u \not\equiv 0, I'(u) = 0, P(u) = 0\}, \\ S_{\text{LES}} &:= \{u \in H^\alpha(\mathbf{R}^N) \mid u \not\equiv 0, I'(u) = 0, P(u) = 0, I(u) = c_{\text{LES}}\}, \end{aligned}$$

we have $S_{\text{LES}} \neq \emptyset$ and $c_{\text{MP}} = c_{\text{LES}} > 0$. Furthermore, for every $v \in S_{\text{LES}}$, there exists a $\gamma_v \in \Gamma$ such that $\|\gamma_v(t)\|_{L^\infty} = \|v\|_{L^\infty}$ for $0 < t \leq 1$ and

$$\max_{0 \leq t \leq 1} I(\gamma_v(t)) = I(v) = c_{\text{LES}}.$$

Remark 1.2. (i) To the author's knowledge, it is not known that every weak solution of (4) satisfies the Pohozaev identity. In Proposition 3.6, we shall show that if a weak solution of (4) is of class C^1 with bounded derivatives, then the Pohozaev identity holds.

(ii) By Theorem 1.1 (ii), when $\alpha > 1/2$ or $f(s)$ is locally Lipschitz, we have

$$\begin{aligned} c_{\text{LES}} &= \inf \{I(u) \mid u \not\equiv 0, I'(u) = 0\}, \\ S_{\text{LES}} &= \{u \in H^\alpha(\mathbf{R}^N) \mid u \not\equiv 0, I'(u) = 0, I(u) = c_{\text{LES}}\}. \end{aligned}$$

Thus c_{LES} and S_{LES} coincide with the least energy and a set of all least energy solutions in usual sense.

(iii) By the simple scaling, we may deal with

$$(m^2 - \Delta)^\alpha u = f(u) \quad \text{in } \mathbf{R}^N$$

where $m > 0$. Indeed, for $m > 0$, $u(x)$ is a solution of (4) if and only if $v(x) := u(m^{-1}x)$ satisfies

$$(1 - \Delta)^\alpha v = m^{-2\alpha} f(v(x)) \quad \text{in } \mathbf{R}^N.$$

Next, we use Theorem 1.1 to obtain a positive solution of (1). For $f(x, s)$, assume that

(F1) $f(x, s) = -V(x)s + g(x, s)$ where $V \in C(\mathbf{R}^N, \mathbf{R})$, $g \in C(\mathbf{R}^N \times \mathbf{R}, \mathbf{R})$ and $g(x, -s) = -g(x, s)$ for every $(x, s) \in \mathbf{R}^N \times \mathbf{R}$.

(F2)

$$-1 < \inf_{x \in \mathbf{R}^N} V(x) \quad \text{and} \quad \lim_{s \rightarrow 0} \sup_{x \in \mathbf{R}^N} \left| \frac{g(x, s)}{s} \right| = 0.$$

(F3)

$$\lim_{|s| \rightarrow \infty} \sup_{x \in \mathbf{R}^N} \frac{|g(x, s)|}{|s|^{2_\alpha^* - 1}} = 0.$$

(F4) There exist $V_\infty > -1$ and $g_\infty(s) \in C(\mathbf{R}, \mathbf{R})$ such that as $|x| \rightarrow \infty$, $V(x) \rightarrow V_\infty$ and $g(x, s) \rightarrow g_\infty(s)$ in $L_{\text{loc}}^\infty(\mathbf{R}^N)$ where $g_\infty(s)$ is locally Lipschitz continuous provided $0 < \alpha \leq 1/2$. Moreover, $0 \leq F(x, s) - F_\infty(s)$ holds for all $x \in \mathbf{R}^N$ and $s \in \mathbf{R}$ where $F(x, s) := \int_0^s f(x, t)dt$, $f_\infty(s) := -V_\infty s + g_\infty(s)$ and $F_\infty(s) := \int_0^s f_\infty(t)dt$.

(F5) There exist $\mu > 2$ and $s_1 > 0$ such that

$$0 < \mu G(x, s) \leq g(x, s)s \quad \text{for each } (x, s) \in \mathbf{R}^N \times \mathbf{R} \setminus \{0\}, \quad \inf_{x \in \mathbf{R}^N} G(x, s_1) > 0$$

$$\text{where } G(x, s) := \int_0^s g(x, t)dt.$$

Under these conditions, we have

Theorem 1.3. *Assume (F1)–(F5). Then (1) admits a positive solution.*

Remark 1.4. (i) In (F4), when $0 < \alpha \leq 1/2$, we assume that $g_\infty(s)$ is local Lipschitz in s , however, not for $g(x, s)$.

(ii) (F5) is mainly used to find a bounded Palais–Smale sequence. If we assume the existence of bounded Palais–Smale sequence at the mountain pass level, we can show the existence of nontrivial solution of (1) in the more general setting. See Proposition 4.3.

(iii) Another way to obtain bounded Palais–Smale sequences is to exploit the Pohozaev identity. When $\alpha = 1$, for example, we refer to [5, 28]. When $0 < \alpha < 1$, we also have the Pohozaev identity (see (46)) and it might be useful to get a bounded Palais–Smale sequence in the case $0 < \alpha < 1$.

Now we compare our result with the previous results. We first consider (4). The most related results are [4, 12, 23, 38]. In these papers, the authors study (4) with $f(s) = |s|^{p-1}s$ or $f(s) = (1 - \mu)s + |s|^{p-1}s$ where $1 < p < 2_\alpha^* - 1$ and $\mu > 0$, and show the existence of least energy solution (or ground state solution) and infinitely many solutions. Clearly, Theorem 1.1 improves these results. Furthermore, in [4, 34], the authors raise a question that one can prove the existence of least energy solution and infinitely many solutions of (4) with general nonlinearity. Theorem 1.1 answers this question. For the fractional Laplacian $(-\Delta)^\alpha$ with general nonlinearity, we refer to the work [11].

On the other hand, for (1), the existence of positive solution is proved in [23, 34]. It is easily checked that Theorem 1.3 is a generalization of the result in [23]. In addition, suppose that $f(x, s) = \lambda b(x)|u|^{p-1}u + c(x)|u|^{q-1}u$, $b(x) \geq \underline{b} = \lim_{|x| \rightarrow \infty} b(x) > 0$ and $c(x) \geq 0 = \lim_{|x| \rightarrow \infty} c(x)$. Then we can apply Theorem 1.3 to get a positive solution of (1) for every $\lambda > 0$ and $1 < p, q < 2_\alpha^* - 1$. Hence, in this case, we find the simpler sufficient condition than that of [34] for the existence of nontrivial solution

Finally, we comment on the proofs of Theorems 1.1 and 1.3. Our arguments are variational and we find critical points of I defined by (5) and

$$J(u) := \frac{1}{2} \|u\|_\alpha^2 - \int_{\mathbf{R}^N} F(x, u(x)) dx \in C^1(H^\alpha(\mathbf{R}^N), \mathbf{R}).$$

To show the existence of critical points of I , we use the arguments in [24, 25] and introduce the augmented functional based on the scaling:

$$\tilde{I}(\theta, u) := I(u(\cdot/e^\theta)) \in C^1(\mathbf{R} \times H^\alpha(\mathbf{R}^N), \mathbf{R}).$$

As already pointed out in [4, 34], $-\Delta$ and $(-\Delta)^\alpha$ are homogenous in scaling, however, $(1-\Delta)^\alpha$ is not. Nevertheless, \tilde{I} still helps us to find bounded Palais–Smale sequences.

Next, we turn to Theorem 1.3. We use the idea of the concentration compactness lemma ([27, 30, 31]) and compare the mountain pass values. First, we treat the general setting and exploiting Theorem 1.1, we prove that it suffices to find a bounded Palais–Smale sequence of J at the mountain pass level. To this end, we observe the behavior of any bounded Palais–Smale sequence of J . After that, we shall prove Theorem 1.3.

This paper is organized as follows. In sections 2 and 3, we introduce the augmented functional $\tilde{I}(\theta, u)$, prove its properties and show Theorem 1.1. Section 4 is devoted to proving Theorem 1.3. In Appendix, we collect some technical lemmas and prove a Brézis-Kato type result (Proposition 3.5).

2. VARIATIONAL SETTING

To prove Theorems 1.1 and 1.3, we employ the variational methods. We first consider (4) and prove Theorem 1.1. In what follows, we always assume (f1)–(f4) and use the following notations: For $K = \mathbf{R}, \mathbf{C}$, $\mathcal{S}(\mathbf{R}^N, K)$ denotes the Schwartz class consisting of K -valued functions. Moreover, set $H_r^\alpha(\mathbf{R}^N) := \{u \in H^\alpha(\mathbf{R}^N) \mid u \text{ is radial}\}$. Recalling (2), we begin with the following lemma.

Lemma 2.1. (i) *The space $H^\alpha(\mathbf{R}^N)$ is a Hilbert space over \mathbf{R} under the following scalar product:*

$$\langle u, v \rangle_\alpha := \int_{\mathbf{R}^N} (4\pi^2 |\xi|^2 + 1)^\alpha \hat{u}(\xi) \overline{\hat{v}(\xi)} d\xi.$$

Notice that $\|u\|_\alpha^2 = \langle u, u \rangle_\alpha$.

(ii) ([29]) *The embedding $H_r^\alpha(\mathbf{R}^N) \subset L^p(\mathbf{R}^N)$ is compact for $2 < p < 2_\alpha^*$.*

(iii) *The functional I in (5) belongs to $C^1(H^\alpha(\mathbf{R}^N), \mathbf{R})$ and*

$$(7) \quad I'(u)\varphi = \int_{\mathbf{R}^N} (1 + 4\pi^2 |\xi|^2)^\alpha \hat{u} \overline{\hat{\varphi}} d\xi - \int_{\mathbf{R}^N} f(u) \varphi dx \quad \text{for all } \varphi \in H^\alpha(\mathbf{R}^N).$$

In particular, if $I'(u) = 0$, then u satisfies (4) in $(\mathcal{S}(\mathbf{R}^N, \mathbf{C}))^*$, that is, for all $\varphi \in \mathcal{S}(\mathbf{R}^N, \mathbf{C})$,

$$(8) \quad \langle (1 - \Delta)^\alpha u, \varphi \rangle = \int_{\mathbf{R}^N} (1 + 4\pi^2 |\xi|^2)^\alpha \hat{u}(\xi) (\mathcal{F}^{-1} \varphi)(\xi) d\xi = \int_{\mathbf{R}^N} f(u) \varphi dx.$$

The same holds true for $I|_{H_r^\alpha(\mathbf{R}^N)}$.

Proof. (i) We only check $\langle u, v \rangle_\alpha \in \mathbf{R}$ for any $u, v \in H^\alpha(\mathbf{R}^N)$. Put

$$G_{2\alpha}(x) := \frac{1}{(4\pi)^\alpha} \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-\pi|x|^2/t} e^{-t/4\pi t^{2\alpha-N}/2} \frac{dt}{t}.$$

Then it is known that (see [35, Chapter V])

$$(9) \quad \begin{aligned} \widehat{G_{2\alpha}}(\xi) &= (4\pi^2 |\xi|^2 + 1)^{-\alpha}, \quad \|G_{2\alpha}\|_{L^1} = 1, \\ 0 \leq G_{2\alpha}(x) &\leq C_0 \left(|x|^{N-2\alpha} \chi_{B_1(0)}(x) + e^{-c_1|x|} \chi_{(B_1(0))^c}(x) \right) \end{aligned}$$

for some $c_1 > 0$ where $B_1(0) := \{x \in \mathbf{R}^N \mid |x| < 1\}$ and χ_A is a characteristic function of A . Moreover, for every $\varphi \in \mathcal{S}(\mathbf{R}^N, \mathbf{R})$, the equation

$$(-\Delta + 1)^\alpha u = \varphi \quad \text{in } \mathbf{R}^N, \quad u \in H^\alpha(\mathbf{R}^N)$$

has a unique solution u expressed as $u = G_{2\alpha} * \varphi \in \mathcal{S}(\mathbf{R}^N, \mathbf{R})$ due to (9). For this u , if $v \in \mathcal{S}(\mathbf{R}^N, \mathbf{R})$, then

$$\begin{aligned} \langle u, v \rangle_\alpha &= \int_{\mathbf{R}^N} (1 + 4\pi |\xi|^2)^\alpha \hat{u}(\xi) \overline{\hat{v}(\xi)} d\xi = \int_{\mathbf{R}^N} (1 + 4\pi |\xi|^2)^\alpha \widehat{G_{2\alpha} * \varphi} \overline{\hat{v}(\xi)} d\xi \\ &= \int_{\mathbf{R}^N} \hat{\varphi} \overline{\hat{v}} d\xi = \int_{\mathbf{R}^N} \varphi v dx \in \mathbf{R}. \end{aligned}$$

Here at the last equality, we used the Plancherel theorem. Therefore, if $\varphi, v \in \mathcal{S}(\mathbf{R}^N, \mathbf{R})$ and $u = G_{2\alpha} * \varphi$, then $\langle u, v \rangle_\alpha \in \mathbf{R}$. Since $\mathcal{S}(\mathbf{R}^N, \mathbf{R})$ is dense in $H^\alpha(\mathbf{R}^N)$, we have

$$\langle G_{2\alpha} * \varphi, v \rangle_\alpha \in \mathbf{R} \quad \text{for all } \varphi \in \mathcal{S}(\mathbf{R}^N, \mathbf{R}), v \in H^\alpha(\mathbf{R}^N).$$

Finally, the map $\varphi \mapsto G_{2\alpha} * \varphi : \mathcal{S}(\mathbf{R}^N, \mathbf{R}) \rightarrow \mathcal{S}(\mathbf{R}^N, \mathbf{R})$ is bijective, by the density argument, we obtain $\langle u, v \rangle_\alpha \in \mathbf{R}$ for every $u, v \in H^\alpha(\mathbf{R}^N)$.

(ii) This is proved in [29].

(iii) Noting $\langle u, v \rangle_\alpha \in \mathbf{R}$ for all $u, v \in H^\alpha(\mathbf{R}^N)$ and (f1)–(f4), it is easy to check $I \in C^1(H^\alpha(\mathbf{R}^N), \mathbf{R})$ and (7). For (8), we see from (7) that

$$\int_{\mathbf{R}^N} (1 + 4\pi^2 |\xi|^2)^\alpha \hat{u} \bar{\hat{\psi}} d\xi = \int_{\mathbf{R}^N} f(u) \bar{\psi} dx \quad \text{for all } \psi \in \mathcal{S}(\mathbf{R}^N, \mathbf{C}).$$

Then setting $\varphi(x) := \overline{\psi(x)}$ and noting $\overline{\mathcal{F}\psi} = \mathcal{F}^{-1}\bar{\psi} = \mathcal{F}^{-1}\varphi$, one observes that (8) holds.

The last assertion follows from the principle of symmetric criticality. See [39]. \square

Hereafter, we shall look for critical points of $I|_{H_\Gamma^\alpha(\mathbf{R}^N)}$. Following the arguments in [24], we first introduce a comparison functional $\bar{I}(u)$, which plays a role to show that the minimax values c_n defined in (18) diverge as $n \rightarrow \infty$. To this end, we modify the nonlinearity $f(s)$. By (f2), choose $\delta_0 > 0$ and $s_1 > 0$ such that

$$(10) \quad sf(s) \leq (1 - 2\delta_0)s^2 \quad \text{for all } |s| \leq s_1.$$

Fixing a $p_0 \in (1, 2_\alpha^* - 1)$, set

$$h(s) := \begin{cases} (f(s) - (1 - \delta_0)s)_+, & \text{if } s \geq 0, \\ -h(-s) & \text{if } s < 0, \end{cases} \quad \bar{h}(s) := \begin{cases} s^{p_0} \sup_{0 < t < s} \frac{h(t)}{t^{p_0}} & \text{if } s > 0, \\ 0 & \text{if } s = 0, \\ -\bar{h}(-s) & \text{if } s < 0 \end{cases}$$

where $a_+ := \max\{0, a\}$. Finally, put $\bar{H}(s) := \int_0^s \bar{h}(t) dt$. Then

Lemma 2.2. (i) $\bar{h} \in C(\mathbf{R})$ is odd, $\bar{h}(s) \geq 0$ for $s \geq 0$, $\bar{h} \not\equiv 0$ and \bar{h} satisfies (f3).

(ii) There exists an $s_2 > 0$ such that $\bar{h}(s) = 0 = \bar{H}(s)$ for all $|s| \leq s_2$. In particular, there is a $C_0 > 0$ such that $|\bar{h}(s)s| + |\bar{H}(s)| \leq C_0 |s|^{2_\alpha^*}$ for each $s \in \mathbf{R}$.

(iii) $0 \leq (p_0 + 1)\bar{H}(s) \leq s\bar{h}(s)$ for any $s \in \mathbf{R}$.

(iv) $F(s) - (1 - \delta_0)s^2/2 \leq \bar{H}(s)$ for every $s \in \mathbf{R}$.

(v) Let $(u_n) \subset H_\Gamma^\alpha(\mathbf{R}^N)$ satisfy $u_n \rightharpoonup u_0$ weakly in $H_\Gamma^\alpha(\mathbf{R}^N)$ and $u_n(x) \rightarrow u_0(x)$ for a.e. $x \in \mathbf{R}^N$. Then

$$\bar{H}(u_n) \rightarrow \bar{H}(u_0) \quad \text{strongly in } L^1(\mathbf{R}^N), \quad \bar{h}(u_n) \rightarrow \bar{h}(u_0) \quad \text{strongly in } L^{2N/(N+2\alpha)}(\mathbf{R}^N).$$

Proof. Since one can prove (i)–(iv) in a similar way to [24, Lemma 2.1 and Corollary 2.2], we omit the details. Now we shall prove (v). Since both of the assertions can be proved in a similar way, we only treat $\bar{h}(u_n) \rightarrow \bar{h}(u_0)$ strongly in $L^{2N/(N+2\alpha)}(\mathbf{R}^N)$. Noting that \bar{h} satisfies (f3) thanks to the assertion (i), for each $\varepsilon > 0$ there exists an $s_\varepsilon > 0$ such that

$$(11) \quad |\bar{h}(s)|^{2N/(N+2\alpha)} \leq \varepsilon |s|^{2_\alpha^*} \quad \text{if } |s| \geq s_\varepsilon.$$

Set $[|u_n| < a] := \{x \in \mathbf{R}^N \mid |u_n(x)| < a\}$, $\chi_{n,\varepsilon}(x) := \chi_{[|u_n| < s_\varepsilon]}(x)$ and $\chi_{0,\varepsilon}(x) := \chi_{[|u_0| < s_\varepsilon]}(x)$. Using $\bar{h}(u_n(x)) = \bar{h}(\chi_{n,\varepsilon}(x)u_n(x)) + \bar{h}((1 - \chi_{n,\varepsilon}(x))u_n(x))$ and writing $v_n(x) := \chi_{n,\varepsilon}(x)u_n(x)$, $v_0(x) := \chi_{0,\varepsilon}(x)u_0(x)$, $w_n(x) := (1 - \chi_{n,\varepsilon}(x))u_n(x)$ and $w_0(x) := (1 - \chi_{0,\varepsilon}(x))u_0(x)$, we have

$$(12) \quad |\bar{h}(u_n) - \bar{h}(u_0)| \leq |\bar{h}(\chi_{0,\varepsilon}v_n) - \bar{h}(v_0)| + |\bar{h}((1 - \chi_{0,\varepsilon})v_n)| + |\bar{h}(w_n) - \bar{h}(w_0)|.$$

Since $w_n(x) \neq 0$ implies $|w_n(x)| \geq s_\varepsilon$, it follows from (11) that

$$(13) \quad \begin{aligned} \sup_{n \geq 1} \int_{\mathbf{R}^N} |\bar{h}(w_n) - \bar{h}(w_0)|^{2N/(N+2\alpha)} dx &\leq C_0 \sup_{n \geq 1} \int_{\mathbf{R}^N} \left(|\bar{h}(w_n)|^{2N/(N+2\alpha)} + |\bar{h}(w_0)|^{2N/(N+2\alpha)} \right) dx \\ &\leq C\varepsilon \sup_{n \geq 1} \left(\|w_n\|_{L^{2_\alpha^*}}^{2_\alpha^*} + \|w_0\|_{L^{2_\alpha^*}}^{2_\alpha^*} \right) \leq C\varepsilon. \end{aligned}$$

Recalling $u_n(x) \rightarrow u_0(x)$ and the definition of $\chi_{0,\varepsilon}(x)$, we observe that

$$\limsup_{n \rightarrow \infty} |(1 - \chi_{0,\varepsilon})(x)v_n(x)| \leq \chi_{[|u_0| = s_\varepsilon]}(x)|s_\varepsilon| \quad \text{for a.e. } x \in \mathbf{R}^N.$$

Hence, using (11) and $\chi_{[|u_0|=s_\varepsilon]}(x) \leq (1 - \chi_{0,\varepsilon}(x))$, we obtain

$$(14) \quad \limsup_{n \rightarrow \infty} \int_{\mathbf{R}^N} |\bar{h}((1 - \chi_{0,\varepsilon})(x)v_n(x))|^{2N/(N+2\alpha)} dx \leq \int_{\mathbf{R}^N} |\bar{h}(\chi_{[|u_0|=s_\varepsilon]}(x)|s_\varepsilon|)|^{2N/(N+2\alpha)} dx \\ \leq \int_{\mathbf{R}^N} |\bar{h}(w_0)|^{2N/(N+2\alpha)} dx \leq C\varepsilon.$$

On the other hand, since $\chi_{0,\varepsilon}(x)v_n(x) \rightarrow v_0(x)$ for a.e. $x \in \mathbf{R}^N$, noting that $u_n \rightarrow u_0$ strongly in $L^p(\mathbf{R}^N)$ for $2 < p < 2_\alpha^*$ due to Lemma 2.1 (ii) and that $|v_n(x)|, |v_0(x)| \leq s_\varepsilon$, we have $\chi_{0,\varepsilon}v_n \rightarrow v_0$ strongly in $L^p(\mathbf{R}^N)$ for $2 < p < \infty$. Thus, by the assertion (ii), it is easily seen that

$$(15) \quad \lim_{n \rightarrow \infty} \int_{\mathbf{R}^N} |\bar{h}(\chi_{0,\varepsilon}v_n) - \bar{h}(v_0)|^{2N/(N+2\alpha)} dx = 0.$$

Collecting (12)–(15), one sees

$$\limsup_{n \rightarrow \infty} \|\bar{h}(u_n) - \bar{h}(u_0)\|_{L^{2N/(N+2\alpha)}}^{2N/(N+2\alpha)} \leq C\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\bar{h}(u_n) \rightarrow \bar{h}(u_0)$ strongly in $L^{2N/(N+2\alpha)}(\mathbf{R}^N)$. \square

Next, from

$$I(u) = \frac{1}{2}\|u\|_\alpha^2 - \frac{1-\delta_0}{2} \int_{\mathbf{R}^N} u^2 dx - \int_{\mathbf{R}^N} F(u) - \frac{1-\delta_0}{2} u^2 dx \\ \geq \frac{\delta_0}{2}\|u\|_\alpha^2 - \int_{\mathbf{R}^N} F(u) - \frac{1-\delta_0}{2} u^2 dx,$$

we define a comparison functional $\bar{I}(u)$ by

$$\bar{I}(u) := \frac{\delta_0}{2}\|u\|_\alpha^2 - \int_{\mathbf{R}^N} \bar{H}(u) dx.$$

Lemma 2.3. (i) *The inequality $\bar{I}(u) \leq I(u)$ holds for any $u \in H^\alpha(\mathbf{R}^N)$. Moreover, there exists a $\rho_0 > 0$ such that*

$$0 < \inf_{\|u\|_\alpha = \rho_0} \bar{I}(u), \quad \bar{I}(u) \geq 0 \quad \text{if } \|u\|_\alpha \leq \rho_0.$$

(ii) *The functional \bar{I} satisfies the Palais–Smale condition.*

(iii) *For each $n \geq 1$, there exists a $\gamma_n \in C(\partial D_n, H_r^\alpha(\mathbf{R}^N))$ such that*

$$\gamma_n(-\sigma) = -\gamma_n(\sigma), \quad I(\gamma_n(\sigma)) < 0 \quad \text{for each } \sigma \in \partial D_n$$

where $D_n := \{\sigma = (\sigma_1, \dots, \sigma_n) \in \mathbf{R}^n \mid |\sigma| \leq 1\}$.

Proof. (i) The inequality $\bar{I}(u) \leq I(u)$ is clear from the definition and Lemma 2.2. Next by Lemma 2.2 (ii), we have $|\bar{H}(s)| \leq C|s|^{2_\alpha^*}$ for all $s \in \mathbf{R}$. Thus it follows from Sobolev's inequality that

$$\bar{I}(u) \geq \frac{\delta_0}{2}\|u\|_\alpha^2 - C \int_{\mathbf{R}^N} |u|^{2_\alpha^*} dx \geq \frac{\delta_0}{2}\|u\|_\alpha^2 - C\|u\|_\alpha^{2_\alpha^*}.$$

Noting $2 < 2_\alpha^*$ and choosing $\rho_0 > 0$ sufficiently small, we get

$$\inf_{\|u\|_\alpha = \rho_0} I(u) > 0, \quad I(u) \geq 0 \quad \text{if } \|u\|_\alpha \leq \rho_0.$$

(ii) Since the nonlinearity \bar{h} satisfies the global Ambrosetti–Rabinowitz condition (Lemma 2.2 (iii)), following the argument in [33] (cf. proof of Theorem 1.3 below) and using Lemma 2.2 (v), we can prove that \bar{I} satisfies the Palais–Smale condition and we omit the details.

(iii) Since $f(s) - s$ satisfies the Berestycki–Lions type conditions (see [7, 8]), as in [8, Theorem 10], we may find a map $\pi_n \in C(\partial D_n, H_r^1(\mathbf{R}^N))$ with the properties

$$0 \notin \pi_n(\partial D_n), \quad \pi_n(-\sigma) = -\pi_n(\sigma), \quad \int_{\mathbf{R}^N} F(\pi_n(\sigma)) - \frac{1}{2}(\pi_n(\sigma))^2 dx \geq 1 \quad \text{for all } \sigma \in \partial D_n.$$

Set $\gamma_n(\sigma)(x) := \pi_n(\sigma)(x/t)$ for $t > 0$. Then for sufficiently large $t > 0$, it follows from $\widehat{\gamma_n(\sigma)}(\xi) = t^N \widehat{\pi_n(\sigma)}(t\xi)$ and the inequality $(1+s)^\alpha \leq 1+s^\alpha$ for $s \geq 0$ that

$$\begin{aligned} I(\gamma_n(\sigma)) &= \frac{t^N}{2} \int_{\mathbf{R}^N} \left(1 + \frac{4\pi^2|\xi|^2}{t^2}\right)^\alpha |\widehat{\pi_n(\sigma)}|^2 d\xi - t^N \int_{\mathbf{R}^N} F(\pi_n(\sigma)) dx \\ &\leq \frac{t^{N-2\alpha}}{2} \int_{\mathbf{R}^N} (4\pi^2|\xi|^2)^\alpha |\widehat{\pi_n(\sigma)}|^2 d\xi - t^N \int_{\mathbf{R}^N} F(\pi_n(\sigma)) - \frac{1}{2} (\pi_n(\sigma))^2 dx \\ &\leq \frac{t^{N-2\alpha}}{2} \int_{\mathbf{R}^N} (4\pi^2|\xi|^2)^\alpha |\widehat{\pi_n(\sigma)}|^2 d\xi - t^N < 0 \quad \text{for all } \sigma \in \partial D_n. \end{aligned}$$

Since $H_r^1(\mathbf{R}^N) \subset H_r^\alpha(\mathbf{R}^N)$, we have $\gamma_n \in C(\partial D_n, H_r^\alpha(\mathbf{R}^N))$ and complete the proof. \square

Remark 2.4. When $n = 1$, we can assume that $\gamma_1(1)(x) \geq 0$ for each $x \in \mathbf{R}^N$, $\gamma_1(1)(|x|) = \gamma_1(1)(x)$ and $r \mapsto \gamma_1(1)(r)$ is piecewise linear and nonincreasing. See [7, 8] (cf. the proof of Proposition 4.1 below).

Now we introduce an auxiliary functional based on the scaling property as in [24, 25]. For this purpose, we set

$$u_\theta(x) := u(e^{-\theta}x).$$

Then we have

$$(16) \quad \widehat{u_\theta}(\xi) = e^{N\theta} \widehat{u}(e^\theta \xi), \quad I(u_\theta) = \frac{e^{N\theta}}{2} \int_{\mathbf{R}^N} \left(1 + 4\pi^2 \frac{|\xi|^2}{e^{2\theta}}\right)^\alpha |\widehat{u}|^2 d\xi - e^{N\theta} \int_{\mathbf{R}^N} F(u) dx.$$

From this, we define $\tilde{I}(\theta, u)$ by

$$\tilde{I}(\theta, u) := \frac{e^{N\theta}}{2} \int_{\mathbf{R}^N} \left(1 + 4\pi^2 \frac{|\xi|^2}{e^{2\theta}}\right)^\alpha |\widehat{u}|^2 d\xi - e^{N\theta} \int_{\mathbf{R}^N} F(u) dx.$$

It is easily seen that $\tilde{I} \in C^1(\mathbf{R} \times H_r^\alpha(\mathbf{R}^N), \mathbf{R})$ and u is a critical point of I if $(0, u)$ is a critical point of \tilde{I} . Furthermore, we see the following relation between \tilde{I} and P (see (6) for the definition of $P(u)$): for all $u \in H^\alpha(\mathbf{R}^N)$ and $\theta \in \mathbf{R}$,

$$\begin{aligned} D_\theta \tilde{I}(\theta, u) &= \frac{N}{2} e^{N\theta} \int_{\mathbf{R}^N} (1 + 4\pi^2 e^{-2\theta} |\xi|^2)^\alpha |\widehat{u}(\xi)|^2 d\xi \\ &\quad - \alpha e^{N\theta} \int_{\mathbf{R}^N} (1 + 4\pi^2 e^{-2\theta} |\xi|^2)^{\alpha-1} e^{-2\theta} 4\pi^2 |\xi|^2 |\widehat{u}(\xi)|^2 d\xi - N e^{N\theta} \int_{\mathbf{R}^N} F(u) dx \\ (17) \quad &= \frac{N}{2} \int_{\mathbf{R}^N} (1 + 4\pi^2 |\xi|^2)^\alpha |\widehat{u_\theta}(\xi)|^2 d\xi \\ &\quad - \alpha \int_{\mathbf{R}^N} (1 + 4\pi^2 |\xi|^2)^{\alpha-1} 4\pi^2 |\xi|^2 |\widehat{u_\theta}(\xi)|^2 d\xi - N \int_{\mathbf{R}^N} F(u) dx \\ &= D_\theta \tilde{I}(0, u_\theta) = \frac{N-2\alpha}{2} \|u_\theta\|_\alpha^2 + \alpha \int_{\mathbf{R}^N} (1 + 4\pi^2 |\xi|^2)^{\alpha-1} |\widehat{u_\theta}|^2 d\xi - N \int_{\mathbf{R}^N} F(u_\theta) dx \\ &= P(u_\theta). \end{aligned}$$

This functional is useful to generate a bounded Palais–Smale sequence (u_k) with $P(u_k) \rightarrow 0$.

Recalling Lemma 2.3, for every $n \geq 1$, we define

$$\begin{aligned} c_n &:= \inf_{\gamma \in \Gamma_n} \max_{\sigma \in D_n} I(\gamma(\sigma)), \quad \tilde{c}_n := \inf_{\tilde{\gamma} \in \tilde{\Gamma}_n} \max_{\sigma \in D_n} \tilde{I}(\tilde{\gamma}(\sigma)), \quad d_n := \inf_{\gamma \in \Gamma_n} \max_{\sigma \in D_n} \bar{I}(\gamma(\sigma)), \\ (18) \quad \Gamma_n &:= \{\gamma \in C(D_n, H_r^\alpha(\mathbf{R}^N)) \mid \gamma(-\sigma) = -\gamma(\sigma), \gamma = \gamma_n \text{ on } \partial D_n\}, \\ \tilde{\Gamma}_n &:= \{\tilde{\gamma}(\sigma) = (\theta(\sigma), \gamma(\sigma)) \in C(D_n, \mathbf{R} \times H_r^\alpha(\mathbf{R}^N)) \mid \\ &\quad \theta(-\sigma) = \theta(\sigma) \text{ for all } \sigma \in D_n, \theta(\sigma) = 0 \text{ on } \partial D_n, \gamma \in \Gamma_n\} \end{aligned}$$

Remark that $\Gamma_n \neq \emptyset$ since $\gamma_{n,0} \in \Gamma_n$ where $\gamma_{n,0}(\sigma) := |\sigma| \gamma_n(\sigma/|\sigma|)$ when $\sigma \in D_n \setminus \{0\}$ and $\gamma_{n,0}(0) := 0$. Furthermore, we have

Lemma 2.5. *For all $n \in \mathbf{N}$, $d_n \leq c_n = \tilde{c}_n$ hold and $d_n \rightarrow \infty$ as $n \rightarrow \infty$.*

Proof. By definition and Lemma 2.3, $d_n \leq c_n$ is clear. Moreover, noting $(0, \gamma) \in \tilde{\Gamma}_n$ for all $\gamma \in \Gamma_n$ and $I(u) = \tilde{I}(0, u)$, we have $\tilde{c}_n \leq c_n$ for all $n \in \mathbf{N}$. On the other hand, let $\tilde{\gamma} = (\theta, \gamma) \in \tilde{\Gamma}_n$ and put $\zeta(\sigma) := \gamma(\sigma)(e^{-\theta(\sigma)}x)$, it follows from (16) that $I(\zeta(\sigma)) = \tilde{I}(\theta(\sigma), \gamma(\sigma))$. From this and $\tilde{\gamma} \in \tilde{\Gamma}$, one can check that $\zeta \in \Gamma_n$ and $c_n \leq \tilde{c}_n$. Thus $c_n = \tilde{c}_n$ holds.

The assertion $d_n \rightarrow \infty$ can be proved in a similar way to [24, Lemma 3.2] (see also [33]) and we skip the details. \square

3. PROOF OF THEOREM 1.1

In this section, we shall prove Theorem 1.1. We first show the assertion (i). To proceed, we first notice that

$$(19) \quad \begin{aligned} D_u \tilde{I}(\theta, u) \varphi &= e^{N\theta} \int_{\mathbf{R}^N} (1 + 4\pi^2 e^{-2\theta} |\xi|^2)^\alpha \widehat{u} \widehat{\varphi} d\xi - e^{N\theta} \int_{\mathbf{R}^N} f(u) \varphi dx \\ &= \int_{\mathbf{R}^N} (1 + 4\pi^2 |\xi|^2)^\alpha \widehat{u_\theta} \widehat{\varphi_\theta} d\xi - \int_{\mathbf{R}^N} f(u_\theta) \varphi_\theta dx = D_u \tilde{I}(0, u_\theta) \varphi_\theta \end{aligned}$$

where $\varphi_\theta(x) := \varphi(e^{-\theta}x)$.

Proposition 3.1. *Suppose that $((\theta_n, u_n))_{n=1}^\infty \subset \mathbf{R} \times H_r^\alpha(\mathbf{R}^N)$ is a Palais–Smale sequence of \tilde{I} , namely, $\tilde{I}(\theta_n, u_n) \rightarrow c \in \mathbf{R}$ and $D_{(\theta, u)} \tilde{I}(\theta_n, u_n) \rightarrow 0$ strongly in $\mathbf{R} \times (H_r^\alpha(\mathbf{R}^N))^*$. Moreover, assume that (θ_n) is bounded. Then (u_n) is bounded in $H_r^\alpha(\mathbf{R}^N)$.*

Proof. Since (θ_n) is bounded, from (17) and (19), we may assume $\theta_n = 0$ by replacing $u_n(x)$ by $u_n(e^{-\theta_n}x)$. Therefore, we have

$$\begin{aligned} c + o(1) &= \tilde{I}(0, u_n) = \frac{1}{2} \|u_n\|_\alpha^2 - \int_{\mathbf{R}^N} F(u_n) dx, \\ o(1) &= D_\theta \tilde{I}(0, u_n) = \frac{N}{2} \|u_n\|_\alpha^2 - N \int_{\mathbf{R}^N} F(u_n) dx - 4\pi^2 \alpha \int_{\mathbf{R}^N} (1 + 4\pi^2 |\xi|^2)^{\alpha-1} |\xi|^2 |\widehat{u_n}|^2 d\xi. \end{aligned}$$

From these, it follows that

$$(20) \quad \left(\int_{\mathbf{R}^N} (1 + 4\pi^2 |\xi|^2)^{\alpha-1} |\xi|^2 |\widehat{u_n}|^2 d\xi \right)_{n=1}^\infty \text{ is bounded.}$$

Thus if

$$(21) \quad (\|u_n\|_{L^2}^2)_{n=1}^\infty = \left(\int_{\mathbf{R}^N} |\widehat{u_n}|^2 d\xi \right)_{n=1}^\infty \text{ is bounded,}$$

then from $(1 + 4\pi^2 |\xi|^2)^{\alpha-1} \leq 1$ and (20) we get

$$\begin{aligned} \|u_n\|_\alpha^2 &= \int_{\mathbf{R}^N} (1 + 4\pi^2 |\xi|^2)^{\alpha-1} (1 + 4\pi^2 |\xi|^2) |\widehat{u_n}(\xi)|^2 d\xi \\ &\leq \int_{\mathbf{R}^N} |\widehat{u_n}|^2 d\xi + 4\pi^2 \int_{\mathbf{R}^N} (1 + 4\pi^2 |\xi|^2)^{\alpha-1} |\xi|^2 |\widehat{u_n}|^2 d\xi \end{aligned}$$

and (u_n) is bounded in $H^\alpha(\mathbf{R}^N)$.

Now we prove (21) by contradiction and suppose that $\tau_n := \|u_n\|_{L^2}^{-2/N} \rightarrow 0$ as $n \rightarrow \infty$. Set $v_n(x) := u_n(\tau_n^{-1}x)$ and observe that

$$(22) \quad \|v_n\|_{L^2}^2 = 1, \quad \int_{\mathbf{R}^N} |\xi|^{2\alpha} |\widehat{v_n}|^2 d\xi = \tau_n^{N-2\alpha} \int_{\mathbf{R}^N} |\xi|^{2\alpha} |\widehat{u_n}|^2 d\xi.$$

Since there exist $C_1, C_2 > 0$ such that

$$C_1 |\xi|^2 \leq (1 + 4\pi^2 |\xi|^2)^{\alpha-1} |\xi|^2 \quad \text{if } |\xi| \leq 1, \quad C_2 (1 + 4\pi^2 |\xi|^2)^\alpha \leq (1 + 4\pi^2 |\xi|^2)^{\alpha-1} |\xi|^2 \quad \text{if } |\xi| \geq 1,$$

we observe from (20) that the quantities

$$(23) \quad \left(\int_{|\xi| \geq 1} (1 + 4\pi^2 |\xi|^2)^\alpha |\widehat{u_n}|^2 d\xi \right)_{n=1}^\infty \text{ and } \left(\int_{|\xi| \leq 1} |\xi|^2 |\widehat{u_n}|^2 d\xi \right)_{n=1}^\infty \text{ are bounded.}$$

Thus we infer that

$$\int_{|\xi| \leq 1} |\widehat{u_n}|^2 d\xi \rightarrow \infty.$$

Now we divide our arguments into three steps. First we show

Step 1: When $N \geq 3$, $v_n \rightharpoonup 0$ weakly in $H_r^\alpha(\mathbf{R}^N)$.

We first see that for $q > 1$, by Hölder's inequality and (23), we obtain

$$\begin{aligned} \int_{|\xi| \leq 1} |\xi|^{2\alpha} |\widehat{u_n}|^2 d\xi &\leq \left(\int_{|\xi| \leq 1} |\xi|^2 |\widehat{u_n}|^2 d\xi \right)^{1/q} \left(\int_{|\xi| \leq 1} |\xi|^{2(\alpha q - 1)/(q-1)} |\widehat{u_n}|^2 d\xi \right)^{1-1/q} \\ &\leq C_q \left(\int_{|\xi| \leq 1} |\xi|^{2(\alpha q - 1)/(q-1)} |\widehat{u_n}|^2 d\xi \right)^{1-1/q}. \end{aligned}$$

Choosing $q = \alpha^{-1} \in (1, \infty)$, one gets

$$\int_{|\xi| \leq 1} |\xi|^{2\alpha} |\widehat{u_n}|^2 d\xi \leq C_\alpha \left(\int_{|\xi| \leq 1} |\widehat{u_n}|^2 d\xi \right)^{1-\alpha} \leq C_\alpha \tau_n^{-N(1-\alpha)}.$$

Thus it follows from (22), (23) and $N \geq 3 > 2\alpha$ that

$$\begin{aligned} \int_{\mathbf{R}^N} |\xi|^{2\alpha} |\widehat{v_n}|^2 d\xi &= \tau_n^{N-2\alpha} \left(\int_{|\xi| \leq 1} + \int_{|\xi| \geq 1} \right) |\xi|^{2\alpha} |\widehat{u_n}|^2 d\xi \\ &\leq C_\alpha \tau_n^{(N-2)\alpha} + \tau_n^{N-2\alpha} \int_{|\xi| \geq 1} |\xi|^{2\alpha} |\widehat{u_n}|^2 d\xi \rightarrow 0. \end{aligned}$$

By Sobolev's inequality $\|u\|_{L^{2^*_\alpha}} \leq C \|\xi|^\alpha \widehat{u}\|_{L^2}$, we obtain $v_n \rightarrow 0$ strongly in $L^{2^*_\alpha}(\mathbf{R}^N)$. Thus $v_n \rightharpoonup 0$ weakly in $H_r^\alpha(\mathbf{R}^N)$.

Step 2: $v_n \rightharpoonup 0$ weakly in $H_r^\alpha(\mathbf{R}^N)$ when $N = 2$.

Let $\zeta_0 \in C_0^\infty(\mathbf{R}^N)$ satisfy $0 \leq \zeta_0 \leq 1$, $\zeta_0(\xi) = 1$ for $|\xi| \leq 1$ and $\zeta_0(\xi) = 0$ for $|\xi| \geq 2$. Set

$$\zeta_n(\xi) := \zeta_0(\tau_n \xi), \quad w_{n,1}(x) := \mathcal{F}^{-1}(\zeta_n(\xi) \widehat{v_n}(\xi)), \quad w_{n,2}(x) := \mathcal{F}^{-1}((1 - \zeta_n(\xi)) \widehat{v_n}(\xi)).$$

Then one sees from the Plancherel theorem and (23) that $v_n = w_{n,1} + w_{n,2}$ and

$$\begin{aligned} \|w_{n,1}\|_{L^2}^2 &= \|\widehat{w_{n,1}}\|_{L^2}^2 = \int_{\mathbf{R}^2} \zeta_n^2 |\widehat{v_n}|^2 d\xi \leq \|v_n\|_{L^2}^2 \leq 1, \\ \|w_{n,2}\|_{L^2}^2 &= \tau_n^2 \int_{|\xi| \geq 1} (1 - \zeta_0(\xi))^2 |\widehat{u_n}|^2 d\xi \rightarrow 0, \\ \int_{\mathbf{R}^N} w_{n,1} \overline{w_{n,2}} dx &= \int_{\mathbf{R}^N} \widehat{w_{n,1}} \overline{\widehat{w_{n,2}}} d\xi = \tau_n^2 \int_{1 \leq |\xi| \leq 2} \zeta_0(\xi) (1 - \zeta_0(\xi)) |\widehat{u_n}|^2 d\xi \rightarrow 0. \end{aligned} \tag{24}$$

We also see that

$$\int_{\mathbf{R}^2} |\xi|^2 |\widehat{w_{n,1}}|^2 d\xi = \tau_n^4 \int_{\mathbf{R}^2} |\xi|^2 \zeta_0(\tau_n \xi)^2 |\widehat{u_n}(\tau_n \xi)|^2 d\xi = \int_{\mathbf{R}^2} |\xi|^2 \zeta_0(\xi)^2 |\widehat{u_n}|^2 d\xi \leq \int_{|\xi| \leq 2} |\xi|^2 |\widehat{u_n}|^2 d\xi.$$

Thus, by (23), $(w_{n,1})$ is bounded in $H_r^1(\mathbf{R}^2)$ and suppose that $w_{n,1} \rightharpoonup w_0$ weakly in $H^1(\mathbf{R}^N)$.

On the other hand, by

$$\int_{\mathbf{R}^2} |\xi|^{2\alpha} |\widehat{w_{n,2}}|^2 d\xi = \tau_n^{2-2\alpha} \int_{\mathbf{R}^2} |\xi|^{2\alpha} (1 - \zeta_0(\xi))^2 |\widehat{u_n}(\xi)|^2 d\xi \leq \tau_n^{2-2\alpha} \int_{|\xi| \geq 1} |\xi|^{2\alpha} |\widehat{u_n}|^2 d\xi \rightarrow 0,$$

it follows from (24) that $w_{n,2} \rightarrow 0$ strongly in $H^\alpha(\mathbf{R}^N)$. Recalling $v_n = w_{n,1} + w_{n,2}$, we get

$$v_n \rightharpoonup w_0 \quad (\in H^1(\mathbf{R}^N)) \quad \text{weakly in } H^\alpha(\mathbf{R}^N). \tag{25}$$

Now let $\varphi \in C_0^\infty(\mathbf{R}^N)$ be radial and set $\varphi_n(x) := \varphi(\tau_n x)$. Noting

$$\|\tau_n^2 \varphi_n\|_\alpha^2 = \tau_n^2 \int_{\mathbf{R}^2} (1 + 4\pi^2 |\xi|^2 \tau_n^2)^\alpha |\widehat{\varphi}|^2 d\xi \rightarrow 0$$

and using $D_u \tilde{I}(u_n, 0) \rightarrow 0$, we infer that

$$\begin{aligned} \int_{\mathbf{R}^2} f(v_n) \varphi dx &= \tau_n^2 \int_{\mathbf{R}^2} f(u_n) \varphi_n dx = \tau_n^2 \langle u_n, \varphi_n \rangle_\alpha + o(\|\tau_n^2 \varphi_n\|_\alpha) \\ &= \int_{\mathbf{R}^2} (1 + 4\pi^2 |\xi|^2 \tau_n^2)^\alpha \widehat{v_n} \widehat{\varphi} d\xi + o(\|\tau_n^2 \varphi_n\|_\alpha). \end{aligned} \tag{26}$$

Since $v_n \rightarrow w_0$ strongly in $L^p(\mathbf{R}^N)$ for $2 < p < 2_\alpha^*$ due to (25) and Lemma 2.1 (ii), by Strauss' lemma ([37, Lemma 2] or [7, Theorem A.I]), (f3) and $\varphi \in C_0^\infty(\mathbf{R}^N)$, one has

$$(27) \quad \int_{\mathbf{R}^2} f(v_n) \varphi dx \rightarrow \int_{\mathbf{R}^2} f(w_0) \varphi dx.$$

On the other hand, since $\widehat{\varphi}$ is rapidly decreasing, it follows that

$$(28) \quad \lim_{n \rightarrow \infty} \int_{\mathbf{R}^2} (1 + 4\pi^2 |\xi|^2 \tau_n^2)^\alpha \widehat{v_n} \widehat{\varphi} d\xi = \int_{\mathbf{R}^2} \widehat{w_0} \widehat{\varphi} d\xi = \int_{\mathbf{R}^2} w_0 \varphi dx.$$

Now by (26)–(28), we finally obtain

$$\int_{\mathbf{R}^2} f(w_0) \varphi dx = \int_{\mathbf{R}^2} w_0 \varphi dx$$

for all radial $\varphi \in C_0^\infty(\mathbf{R}^2)$, which yields

$$(29) \quad f(w_0) - w_0 \equiv 0 \quad \text{in } \mathbf{R}^2.$$

On the other hand, from (f2), one may find some $s_1 > 0$ such that $s(f(s) - s) < 0$ for all $|s| \leq s_1$ with $s \neq 0$. Thus by (29), $w_0 \in H_r^1(\mathbf{R}^N) \subset C(\mathbf{R}^N \setminus \{0\})$ and $w_0(x) \rightarrow 0$ as $|x| \rightarrow \infty$, we conclude that $w_0 \equiv 0$ and Step 2 holds due to (25).

Step 3: Conclusion

Now we derive a contradiction and conclude that (21) holds. Since (v_n) is bounded in $H^\alpha(\mathbf{R}^N)$ from Steps 1 and 2, we first remark that

$$\|\tau_n^N u_n\|_\alpha^2 = \int_{\mathbf{R}^N} (1 + 4\pi^2 |\xi|^2)^\alpha |\widehat{v_n}(\tau_n^{-1} \xi)|^2 d\xi = \tau_n^N \int_{\mathbf{R}^N} (1 + 4\pi^2 |\xi|^2 \tau_n^2)^\alpha |\widehat{v_n}|^2 d\xi \rightarrow 0.$$

Let $\delta_0 > 0$ and $s_1 > 0$ be constants appearing in (10). It follows from $1 = \|v_n\|_{L^2}^2 \leq \tau_n^N \|u_n\|_\alpha^2$ and $D_u \tilde{I}(0, u_n) \rightarrow 0$ that

$$(30) \quad \begin{aligned} \delta_0 &= \delta_0 \|v_n\|_{L^2}^2 \leq \tau_n^N \|u_n\|_\alpha^2 - (1 - \delta_0) \|v_n\|_{L^2}^2 \\ &= \int_{\mathbf{R}^N} f(u_n) \tau_n^N u_n dx + o(1) - (1 - \delta_0) \|v_n\|_{L^2}^2 \\ &= \int_{\mathbf{R}^N} f(v_n) v_n - (1 - \delta_0) v_n^2 dx + o(1) \\ &\leq \int_{\mathbf{R}^N} (f(v_n) v_n - (1 - \delta_0) v_n^2)_+ dx + o(1). \end{aligned}$$

By (10), we observe $(f(s)s - (1 - \delta_0)s^2)_+ = 0$ for $|s| \leq s_1$. Hence, arguing as in the proof of Lemma 2.2 (v), it follows from (30) and $v_n \rightarrow 0$ weakly in $H_r^\alpha(\mathbf{R}^N)$ due to Steps 1 and 2 that

$$\delta_0 \leq \int_{\mathbf{R}^N} (f(v_n) v_n - (1 - \delta_0) v_n^2)_+ dx + o(1) \rightarrow 0,$$

which is a contradiction. Thus (21) holds and we complete the proof. \square

Now we prove the existence of critical points of I which satisfy the Pohozaev identity $P(u) = 0$ and correspond to c_n in (18).

Proposition 3.2. *There exist a sequence $(u_n) \subset H_r^\alpha(\mathbf{R}^N)$ such that $I'(u_n) = 0$, $I(u_n) = c_n$ and $P(u_n) = 0$. Especially, (4) has infinitely many solutions satisfying the Pohozaev identity.*

Proof. By Lemma 2.5, we have $c_n = \tilde{c}_n$. Hence, there exists a sequence $(\gamma_{n,k}) \subset \Gamma_n$ such that

$$\max_{\sigma \in D_n} \tilde{I}(0, \gamma_{n,k}(\sigma)) = \max_{\sigma \in D_n} I(\gamma_{n,k}(\sigma)) \rightarrow \tilde{c}_n.$$

Applying Ekeland's variational principle to $(\gamma_{n,k})$ and \tilde{I} , there exist $(\theta_{n,k}, u_{n,k}) \in \mathbf{R} \times H_r^\alpha(\mathbf{R}^N)$ such that

$$\text{dist}((\theta_{n,k}, u_{n,k}), \{0\} \times \gamma_{n,k}(D_n)) \rightarrow 0, \quad \tilde{I}(\theta_{n,k}, u_{n,k}) \rightarrow \tilde{c}_n, \quad D_{(\theta,u)} \tilde{I}(\theta_{n,k}, u_{n,k}) \rightarrow 0.$$

In particular, $\theta_{n,k} \rightarrow 0$. Thus by Proposition 3.1, $(u_{n,k})_k$ is bounded in $H_r^\alpha(\mathbf{R}^N)$.

Now assume $u_{n,k} \rightharpoonup u_{n,0}$ weakly in $H_r^\alpha(\mathbf{R}^N)$ and $u_{n,k} \rightarrow u_{n,0}$ strongly in $L^p(\mathbf{R}^N)$ for $2 < p < 2_\alpha^*$. Let $\delta_0, s_1 > 0$ be constants in (10). By (f4), $\sup_{s \in [0, \infty)} (f(s) - (1 - \delta_0)s) > 0$. Since $f(s)$ is odd, we may find an $s_+ > 0$ satisfying

$$f(\pm s_+) - (1 - \delta_0)(\pm s_+) = 0, \quad f(s) - (1 - \delta_0)s \neq 0 \quad \text{for } s \in (-s_+, s_+) \setminus \{0\}.$$

Set $f_1(s) := f(s) - (1 - \delta_0)s$ if $s \in [-s_+, s_+]$ and $f_1(s) := 0$ otherwise, and $f_2(s) := f(s) - (1 - \delta_0)s - f_1(s)$. Remark that $sf_1(s) \leq 0$ for all $s \in \mathbf{R}$ and $f_2(s)s = 0$ for all $s \in [-s_+, s_+]$.

By $D_u \tilde{I}(u_{n,k}) \rightarrow 0$, one can check $I'(u_{n,0}) = 0$. Moreover, note that a norm defined by

$$\|u\|^2 := \|u\|_\alpha^2 - (1 - \delta_0)\|u\|_{L^2}^2$$

is equivalent to $\|\cdot\|_\alpha$. Thus, arguing as in Step 3 of Proposition 3.1 (see also the proof of Lemma 2.2 (v)), from the boundedness of $(u_{n,k})$, $D_u \tilde{I}(\theta_{n,k}, u_{n,k}) \rightarrow 0$, $\theta_{n,k} \rightarrow 0$, Fatou's lemma to $f_1(u_{n,k})u_{n,k}$, properties of $f_i(s)$ ($i = 1, 2$), $I'(u_{n,0}) = 0$ and the weak convergence of $(u_{n,k})$, we observe that

$$\begin{aligned} \limsup_{k \rightarrow \infty} \|u_{n,k}\|^2 &= \limsup_{k \rightarrow \infty} \int_{\mathbf{R}^N} f(u_{n,k})u_{n,k} - (1 - \delta_0)u_{n,k}^2 dx \\ &\leq \limsup_{k \rightarrow \infty} \int_{\mathbf{R}^N} f_1(u_{n,k})u_{n,k} dx + \limsup_{k \rightarrow \infty} \int_{\mathbf{R}^N} f_2(u_{n,k})u_{n,k} dx \\ &\leq \int_{\mathbf{R}^N} f_1(u_{n,0})u_{n,0} dx + \int_{\mathbf{R}^N} f_2(u_{n,0})u_{n,0} dx \\ &= \int_{\mathbf{R}^N} f(u_{n,0})u_{n,0} - (1 - \delta_0)u_{n,0}^2 dx = \|u_{n,0}\|^2 \leq \liminf_{k \rightarrow \infty} \|u_{n,k}\|^2. \end{aligned}$$

This implies that $u_{n,k} \rightarrow u_{n,0}$ strongly in $H_r^\alpha(\mathbf{R}^N)$. Therefore, $I(u_{n,k}) \rightarrow \tilde{c}_n = c_n = I(u_{n,0})$ and $I'(u_{n,0}) = 0$. Moreover, recalling (17), we have

$$\lim_{k \rightarrow \infty} D_\theta \tilde{I}(\theta_{n,k}, u_{n,k}) \rightarrow 0 = D_\theta \tilde{I}(0, u_{n,0}) = P(u_{n,0}).$$

This completes the proof. \square

By Proposition 3.2, a set

$$S := \{u \in H^\alpha(\mathbf{R}^N) \mid u \not\equiv 0, I'(u) = 0, P(u) = 0\}$$

is not empty. Moreover, we have $c_{\text{LES}} \leq c_1$. Next we show

Proposition 3.3. *For every $u \in H^\alpha(\mathbf{R}^N)$ with $P(u) = 0$ and $u \not\equiv 0$, a path $\gamma_u(t) := u(x/t)$ for $t > 0$ and $\gamma_u(0) := 0$ satisfies*

$$\gamma_u \in C([0, \infty), H^\alpha(\mathbf{R}^N)), \quad I(u) > I(\gamma_u(t)) \quad \text{for any } t \neq 1, \quad I(\gamma_u(t)) \rightarrow -\infty \quad \text{as } t \rightarrow \infty.$$

Proof. For $t > 0$, one sees

$$\|\gamma_u(t)\|_{L^2}^2 = t^N \|u\|_{L^2}^2, \quad \|\xi|^\alpha \widehat{\gamma_u(t)}\|_{L^2}^2 = t^{N-2\alpha} \int_{\mathbf{R}^N} |\xi|^{2\alpha} |\widehat{u}|^2 d\xi.$$

Thus $\gamma_u \in C([0, \infty), H^\alpha(\mathbf{R}^N))$. Furthermore, it follows from $I(\gamma_u(t)) = \tilde{I}(\log t, u)$ and (17) that

$$\begin{aligned} \frac{d}{dt} I(\gamma_u(t)) &= D_\theta \tilde{I}(\log t, u) \frac{1}{t} \\ &= t^{N-1} \left\{ \frac{N}{2} \int_{\mathbf{R}^N} \left(1 + 4\pi^2 \frac{|\xi|^2}{t^2}\right)^\alpha |\widehat{u}|^2 d\xi - N \int_{\mathbf{R}^N} F(u) dx \right. \\ &\quad \left. - \alpha \int_{\mathbf{R}^N} \left(1 + 4\pi^2 \frac{|\xi|^2}{t^2}\right)^{\alpha-1} 4\pi^2 \frac{|\xi|^2}{t^2} |\widehat{u}|^2 d\xi \right\} \\ &= t^{N-1} \left\{ \frac{N-2\alpha}{2} \int_{\mathbf{R}^N} \left(1 + 4\pi^2 \frac{|\xi|^2}{t^2}\right)^\alpha |\widehat{u}|^2 d\xi - N \int_{\mathbf{R}^N} F(u) dx \right. \\ &\quad \left. + \alpha \int_{\mathbf{R}^N} \left(1 + 4\pi^2 \frac{|\xi|^2}{t^2}\right)^{\alpha-1} |\widehat{u}|^2 d\xi \right\}. \end{aligned}$$

Now set

$$g(t) := \frac{N-2\alpha}{2} \int_{\mathbf{R}^N} \left(1 + 4\pi^2 \frac{|\xi|^2}{t^2}\right)^\alpha |\widehat{u}|^2 d\xi - N \int_{\mathbf{R}^N} F(u) dx + \alpha \int_{\mathbf{R}^N} \left(1 + 4\pi^2 \frac{|\xi|^2}{t^2}\right)^{\alpha-1} |\widehat{u}|^2 d\xi.$$

Then we get

$$\begin{aligned}
g'(t) &= \frac{N-2\alpha}{2} \alpha \int_{\mathbf{R}^N} \left(1 + 4\pi^2 \frac{|\xi|^2}{t^2}\right)^{\alpha-1} (-2) \frac{4\pi^2 |\xi|^2}{t^3} |\widehat{u}|^2 d\xi \\
&\quad + \alpha(\alpha-1) \int_{\mathbf{R}^N} \left(1 + 4\pi^2 \frac{|\xi|^2}{t^2}\right)^{\alpha-2} (-2) \frac{4\pi^2 |\xi|^2}{t^3} |\widehat{u}|^2 d\xi \\
&= -2\alpha \int_{\mathbf{R}^N} \left(1 + 4\pi^2 \frac{|\xi|^2}{t^2}\right)^{\alpha-2} \frac{4\pi^2 |\xi|^2}{t^3} |\widehat{u}|^2 \left\{ \frac{N-2\alpha}{2} \left(1 + 4\pi^2 \frac{|\xi|^2}{t^2}\right) + (\alpha-1) \right\} d\xi \\
&= -2\alpha \int_{\mathbf{R}^N} \left(1 + 4\pi^2 \frac{|\xi|^2}{t^2}\right)^{\alpha-2} \frac{4\pi^2 |\xi|^2}{t^3} |\widehat{u}|^2 \left\{ \left(\frac{N}{2} - 1\right) + \frac{2\pi^2(N-2\alpha)|\xi|^2}{t^2} \right\} d\xi.
\end{aligned}$$

Since $N \geq 2$ and $N > 2\alpha$, one has $g'(t) < 0$ for all $t > 0$. Noting that $dI(\gamma_u(t))/dt = t^{N-1}g(t)$ and that $P(u) = 0$ is equivalent to $g(1) = 0$, we see that

$$\frac{d}{dt}I(\gamma_u(t)) > 0 \quad \text{if } 0 < t < 1, \quad \frac{d}{dt}I(\gamma_u(t)) < 0 \quad \text{if } 1 < t,$$

which implies that $I(\gamma_u(t))$ has a unique maximum at $t = 1$. From the monotonicity of $g(t)$ and $g(1) = 0$, it is clear that $I(\gamma_u(t)) \rightarrow -\infty$ as $t \rightarrow \infty$ and Proposition 3.3 holds. \square

Before proceeding to a proof of $c_1 = c_{\text{LES}}$, we define the following quantities:

$$\begin{aligned}
c_{\text{MP},r} &:= \inf_{\gamma \in \Gamma_r} \max_{0 \leq t \leq 1} I(\gamma(t)), \quad \Gamma_r := \{\gamma \in C([0,1], H^\alpha(\mathbf{R}^N)) \mid \gamma(0) = 0, I(\gamma(1)) < 0\}, \\
c_{\text{MP}} &:= \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I(\gamma(t)), \quad \Gamma := \{\gamma \in C([0,1], H^\alpha(\mathbf{R}^N)) \mid \gamma(0) = 0, I(\gamma(1)) < 0\}. \\
\tilde{c}_{\text{MP},r} &:= \inf_{\tilde{\gamma} \in \tilde{\Gamma}_r} \max_{0 \leq t \leq 1} \tilde{I}(\tilde{\gamma}(t)), \\
\tilde{\Gamma}_r &:= \{\tilde{\gamma} = (\theta, \gamma) \in C([0,1], \mathbf{R} \times H^\alpha(\mathbf{R}^N)) \mid \gamma \in \Gamma_r, \theta(0) = 0 = \theta(1)\}.
\end{aligned} \tag{31}$$

Then we show

Lemma 3.4. $0 < c_{\text{MP}} = c_{\text{MP},r} = c_1 = c_{\text{LES}} = \tilde{c}_{\text{MP},r}$.

Proof. As in the proof of Lemma 2.5, one sees $\tilde{c}_{\text{MP},r} = c_{\text{MP},r}$. Moreover, from the definition, $c_{\text{MP}} \leq c_{\text{MP},r} \leq c_1$. In addition, by Propositions 3.2 and 3.3, we know $c_{\text{MP}} \leq c_{\text{LES}} \leq c_1$. Thus it is sufficient to prove $c_1 \leq c_{\text{MP}}$.

We first claim that

$$(32) \quad c_{\text{MP}} = d := \inf_{\eta \in \bar{\Gamma}} \max_{0 \leq t \leq 1} I(\eta(t)), \quad \bar{\Gamma} := \{\eta \in C([0,1], H^\alpha(\mathbf{R}^N)) \mid \eta(0) = 0, \eta(1) = \gamma_1(1)\}$$

where γ_1 appears in Lemma 2.3. From the definition of Γ and $I(\gamma_1(1)) < 0$, we have $c_{\text{MP}} \leq d$. For the opposite inequality $d \leq c_{\text{MP}}$, it is enough to show that $[I < 0] := \{u \in H^\alpha(\mathbf{R}^N) \mid I(u) < 0\}$ is path-connected in $H^\alpha(\mathbf{R}^N)$. A similar claim is proved in [24] for the case $\alpha = 1$ and we use the same argument.

Let $u_1, u_2 \in [I < 0]$. Since $C_0^\infty(\mathbf{R}^N)$ is dense in $H^\alpha(\mathbf{R}^N)$, we may assume $u_1, u_2 \in C_0^\infty(\mathbf{R}^N)$. For u_i , we consider the path $\gamma_i(t) = \gamma_{u_i}(t)$ appearing in Proposition 3.3. From the computations in the proof of Proposition 3.3, we observe that $dI(\gamma_i(t))/dt > 0$ if $0 < t \ll 1$. Since $I(\gamma_i(0)) = 0 > I(\gamma_i(1)) = I(u_i)$, there are maximum points $t_i \in (0,1)$ of $I(\gamma_i(t))$ with $I(\gamma_i(t_i)) > 0$. At those points, we have $dI(\gamma_i(t))/dt|_{t=t_i} = 0$, which yields $P(\gamma_i(t_i)) = 0$. Hence, by Proposition 3.3, we observe that $t \mapsto I(\gamma_i(t)) : (t_i, \infty) \rightarrow \mathbf{R}$ is strictly decreasing and $I(\gamma_i(t)) \rightarrow -\infty$ as $t \rightarrow \infty$. Thus choose $t_0 > 1$ so large that

$$I(\gamma_i(t_0)) < -2 \max \left\{ \max_{0 \leq s \leq 1} I(su_1), \max_{0 \leq s \leq 1} I(su_2) \right\} < 0.$$

Noting that $\gamma_i(t_0)(x) = u_i(x/t_0)$, $u_i \in C_0^\infty(\mathbf{R}^N)$ and

$$\langle u_1(x/t_0), su_2(x - R\mathbf{e}_1) \rangle_\alpha \rightarrow 0,$$

$$\int_{\mathbf{R}^N} F(u_1(x/t_0) + su_2(x - R\mathbf{e}_1)) dx \rightarrow \int_{\mathbf{R}^N} F(u_1/t_0) dx + \int_{\mathbf{R}^N} F(su_2(x)) dx$$

uniformly with respect to $s \in [0,1]$ as $R \rightarrow \infty$ where $\mathbf{e}_1 = (1, 0, \dots, 0)$, it follows from the choice of t_0 that as $R \rightarrow \infty$

$$\max_{0 \leq s \leq 1} I(\gamma_1(t_0) + su_2(x - R\mathbf{e}_1)) \rightarrow I(\gamma_1(t_0)) + \max_{0 \leq s \leq 1} I(su_2) < -\max_{0 \leq s \leq 1} I(s\gamma_2(t_0)) < 0.$$

Hence, choosing $R_0 > 0$ so large, we have

$$\begin{aligned} \text{supp } \gamma_1(t) \cap \text{supp } u_2(\cdot - R_0 \mathbf{e}_1) &= \emptyset \quad \text{for } 1 \leq t \leq t_0, \quad \max_{0 \leq s \leq 1} I(\gamma_1(t_0) + su_2(\cdot - R_0 \mathbf{e}_1)) < 0, \\ I(\gamma_1(t) + u_2(\cdot - R_0 \mathbf{e}_1)) &= I(\gamma_1(t)) + I(u_2) < 0 \quad \text{for } 1 \leq t \leq t_0. \end{aligned}$$

Through the paths $t \mapsto \gamma_1(t)$ ($t \in [1, t_0]$), $s \mapsto \gamma_1(t_0)(x) + su_2(x - R_0 \mathbf{e}_1)$ ($s \in [0, 1]$) and $\theta \mapsto \gamma_1(t_0 - \theta)(x) + u_2(x - R_0 \mathbf{e}_1)$ ($\theta \in [0, t_0 - 1]$), we can connect u_1 and $u_1(x) + u_2(x - R_0 \mathbf{e}_1)$ in $[I < 0]$. In a similar fashion, we see that there is a path between u_2 and $u_1(x) + u_2(x - R_0 \mathbf{e}_1)$ in $[I < 0]$. Therefore, $[I < 0]$ is path-connected in $H^\alpha(\mathbf{R}^N)$ and $c_{\text{MP}} = d$ follows.

Next, since $I(u) = I(-u)$, $\gamma_1(1) \in H_r^\alpha(\mathbf{R}^N)$ and $\gamma(-t) = -\gamma(t)$ holds for any $\gamma \in \Gamma_1$, it suffices to prove

$$(33) \quad d = \inf_{\eta \in \bar{\Gamma}_r} \max_{0 \leq t \leq 1} I(\eta(t)) = c_1, \quad \bar{\Gamma}_r := \{\eta \in C([0, 1], H_r^\alpha(\mathbf{R}^N)) \mid \eta(0) = 0, \eta(1) = \gamma_1(1)\}.$$

By definition, one has $d \leq c_1$.

On the other hand, let $\eta \in \bar{\Gamma}$. Since $F(s)$ is even, we have $\int_{\mathbf{R}^N} F(u) dx = \int_{\mathbf{R}^N} F(|u|) dx$ for each $u \in H^\alpha(\mathbf{R}^N)$. Moreover, noting the following inequality (see Lemma A.3)

$$(34) \quad \| |u| \|_\alpha \leq \|u\|_\alpha \quad \text{for all } u \in H^\alpha(\mathbf{R}^N),$$

we observe that $\zeta(t)(x) := |\eta(t)(x)|$ satisfies

$$I(\zeta(t)) \leq I(\eta(t)) \quad \text{for all } t \in [0, 1], \quad \zeta \in C([0, 1], H^\alpha(\mathbf{R}^N)).$$

Recalling Remark 2.4, we have $\zeta \in \bar{\Gamma}$.

Now set $\tilde{\zeta}(t)(x) := (\zeta(t))^*(x)$ where u^* denotes the Schwarz symmetrization of u . By [1, Theorem 9.2], we have $\tilde{\zeta} \in C([0, 1], H_r^\alpha(\mathbf{R}^N))$ and $\int_{\mathbf{R}^N} F(\tilde{\zeta}(t)) dx = \int_{\mathbf{R}^N} F(\zeta(t)) dx$. Moreover, from Remark 2.4, it follows that $(\gamma_1(1))^* = \gamma_1(1)$, which yields $\tilde{\zeta} \in \bar{\Gamma}_r$. Since $\|\tilde{\zeta}(t)\|_\alpha \leq \|\zeta(t)\|_\alpha$ holds thanks to [34, Proposition 4], we see that

$$c_1 \leq \max_{0 \leq t \leq 1} I(\tilde{\zeta}(t)) \leq \max_{0 \leq t \leq 1} I(\zeta(t)) \leq \max_{0 \leq t \leq 1} I(\eta(t)).$$

Since η is any element of $\bar{\Gamma}$, one sees $c_1 \leq d$.

Therefore, from (32) and (33), we get $c_{\text{MP}} = d = c_1$ and this completes the proof. \square

Finally we shall show that if either $\alpha > 1/2$ or $f(s)$ is locally Lipschitz, then every weak solution of (4) satisfies $P(u) = 0$. To this end, we use the following Brézis–Kato type result [9]:

Proposition 3.5. *Assume that $u \in H^\alpha(\mathbf{R}^N)$ is a weak solution of*

$$(35) \quad (1 - \Delta)^\alpha u - a(x)u = 0 \quad \text{in } \mathbf{R}^N$$

where $a(x)$ satisfies

$$(36) \quad |a(x)| \leq C_0(1 + A(x)) \quad \text{for a.e. } x \in \mathbf{R}^N, \quad A \in L^{N/(2\alpha)}(\mathbf{R}^N).$$

Then $u \in L^p(\mathbf{R}^N)$ for all $p \in [2, \infty)$.

We give a sketch of proof for Proposition 3.5 in Appendix. For related results, see [21]. Using Proposition 3.5, we shall prove

Proposition 3.6. (i) *Let $u \in H^\alpha(\mathbf{R}^N)$ be a weak solution of (4). Then $u \in C_b^\beta(\mathbf{R}^N)$ for every $\beta \in (0, 2\alpha)$ and u decays faster than any polynomial, i.e., for any $k \in \mathbf{N}$ there exists a $c_k > 0$ such that $|u(x)| \leq c_k(1 + |x|)^{-k}$ for all $x \in \mathbf{R}^N$. Here*

$$\begin{aligned} C_b^\beta(\mathbf{R}^N) &:= \left\{ u \in C(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N) \mid [u]_{C^\beta} := \sup_{x, y \in \mathbf{R}^N, x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\beta} < \infty \right\} \quad \text{if } \beta < 1, \\ C_b^1(\mathbf{R}^N) &:= \{u \in C^1(\mathbf{R}^N) \mid u, \nabla u \in L^\infty(\mathbf{R}^N)\}, \\ C_b^\beta(\mathbf{R}^N) &:= \{u \in C_b^1(\mathbf{R}^N) \mid \nabla u \in C_b^{\beta-1}(\mathbf{R}^N)\} \quad \text{if } 1 < \beta < 2. \end{aligned}$$

(ii) *In addition to (f1)–(f4), assume that $f(s)$ is locally Lipschitz continuous and $0 < \alpha \leq 1/2$. Then $u \in C_b^{1+\beta}(\mathbf{R}^N)$ for all $\beta \in (0, 2\alpha)$.*

(iii) *Assume that $u \in H_r^\alpha(\mathbf{R}^N) \cap C_b^1(\mathbf{R}^N)$ is a weak solution of (4). Then $P(u) = 0$.*

A proof of (iii) below is essentially due to [23]. See also [34].

Proof. (i) Let $u \in H^\alpha(\mathbf{R}^N)$ be a weak solution of (4) and set $a(x) := f(u(x))/u(x)$. From (f2) and (f3), it follows that

$$|a(x)| \leq C(1 + |u(x)|^{2_\alpha^* - 2}) \quad \text{for a.e. } x \in \mathbf{R}^N.$$

Noting that u is a weak solution of $(1 - \Delta)^\alpha u - a(x)u = 0$ in \mathbf{R}^N , Proposition 3.5 yields $u \in L^p(\mathbf{R}^N)$ for all $p \in [2, \infty)$. Using (f2) and (f3) again, we observe that $f(u(x)) \in L^p(\mathbf{R}^N)$ for any $p \in [2, \infty)$. Thus, recalling the argument in Lemma 2.1, we have

$$u = G_{2\alpha} * f(u), \quad u \in \mathcal{L}_{2\alpha}^p := \{G_{2\alpha} * g \mid g \in L^p(\mathbf{R}^N)\} \quad \text{for all } p \in [2, \infty).$$

Since $\mathcal{L}_{2\alpha}^p \subset W^{2\alpha, p}(\mathbf{R}^N)$ (see [35]), Sobolev's inequality yields $u \in C_b^\beta(\mathbf{R}^N)$ for each $\beta \in (0, 2\alpha)$. Hence, the first assertion in (i) holds.

For the decay estimate, let $\delta_0 > 0$ be a constant appearing in (10) and $v \in H^\alpha(\mathbf{R}^N)$ a unique weak solution of

$$(1 - \Delta)^\alpha v - (1 - \delta_0)v = (f(u) - (1 - \delta_0)u)_+ =: g(x) \quad \text{in } \mathbf{R}^N.$$

Since $u \in C_b^\beta(\mathbf{R}^N)$, we have $u(x) \rightarrow 0$ as $|x| \rightarrow \infty$ and $g(x) \in L^\infty(\mathbf{R}^N)$ has compact support. Hence, by Proposition A.1 (ii), for any $k \in \mathbf{N}$, v satisfies $v(x) \leq c_k(1 + |x|)^{-k}$ for $x \in \mathbf{R}^N$. Moreover, rewriting (4) as

$$(1 - \Delta)^\alpha u - (1 - \delta_0)u = f(u) - (1 - \delta_0)u \quad \text{in } \mathbf{R}^N,$$

Proposition A.1 (iii) asserts $u(x) \leq v(x) \leq c_k(1 + |x|)^{-k}$ for all $x \in \mathbf{R}^N$.

On the other hand, let w be a solution of

$$(1 - \Delta)^\alpha w - (1 - \delta_0)w = (f(u) - (1 - \delta_0)u)_- \quad \text{in } \mathbf{R}^N.$$

Since

$$(1 - \Delta)^\alpha(-u) - (1 - \delta_0)(-u) = -(f(u) - (1 - \delta_0)u) \leq (1 - \Delta)^\alpha w - (1 - \delta_0)w \quad \text{in } \mathbf{R}^N,$$

using Proposition A.1 again, we get $-u(x) \leq w(x) \leq c_k(1 + |x|)^{-k}$ for $x \in \mathbf{R}^N$. Thus (i) holds.

(ii) Let $f(s)$ be locally Lipschitz and $0 < \alpha \leq 1/2$. Since $u \in C_b^\beta(\mathbf{R}^N)$ for $0 < \beta < 2\alpha$ by (i), we have

$$|f(u(x_1)) - f(u(x_2))| \leq L|u(x_1) - u(x_2)| \leq L[u]_{C^\beta}|x_1 - x_2|^\beta.$$

Thus $f(u(x)) \in C_b^\beta(\mathbf{R}^N)$. Since $u = G_{2\alpha} * f(u)$ and $C_b^{\beta+2\alpha}(\mathbf{R}^N) = G_{2\alpha} * C_b^\beta(\mathbf{R}^N)$ holds by [35, Theorem 4 in §5 of Chapter V], applying the bootstrap argument, we can check that $u \in C_b^\beta(\mathbf{R}^N)$ for all $\beta < 1 + 2\alpha$.

(iii) We follow the argument in [23, 34]. Let $u \in H^\alpha(\mathbf{R}^N) \cap C_b^1(\mathbf{R}^N)$ be a weak solution of (4). For a mollifier (ρ_ε) , set $u_\varepsilon(x) := u * \rho_\varepsilon$. Thanks to the decay estimate of u , we observe that $u_\varepsilon \in \mathcal{S}(\mathbf{R}^N, \mathbf{R})$. Thus,

$$(37) \quad \langle u, x \cdot \nabla u_\varepsilon \rangle_\alpha = \int_{\mathbf{R}^N} f(u) x \cdot \nabla u_\varepsilon dx.$$

Noting

$$\begin{aligned} \langle u, x \cdot \nabla u_\varepsilon \rangle_\alpha &= \int_{\mathbf{R}^N} \widehat{u}(\xi) \overline{\widehat{(1 + 4\pi^2|\xi|^2)^\alpha x \cdot \nabla u_\varepsilon}} d\xi = \int_{\mathbf{R}^N} u(x) (1 - \Delta)^\alpha (x \cdot \nabla u_\varepsilon) dx, \\ (1 - \Delta)^\alpha u_\varepsilon &= (1 - \Delta)^\alpha (\rho_\varepsilon * u) = \rho_\varepsilon * (1 - \Delta)^\alpha u = \rho_\varepsilon * f(u), \end{aligned}$$

and using ([23, Proposition 5.1])

$$(1 - \Delta)^\alpha (x \cdot \nabla u_\varepsilon) = x \cdot \nabla [(1 - \Delta)^\alpha u_\varepsilon] + 2\alpha(1 - \Delta)^\alpha u_\varepsilon - 2\alpha(1 - \Delta)^{\alpha-1} u_\varepsilon,$$

we obtain

$$\begin{aligned} (38) \quad \langle u, x \cdot \nabla u_\varepsilon \rangle_\alpha &= \int_{\mathbf{R}^N} u(x) (1 - \Delta)^\alpha (x \cdot \nabla u_\varepsilon) \\ &= \int_{\mathbf{R}^N} u(x) [x \cdot \nabla (\rho_\varepsilon * f(u)) + 2\alpha \rho_\varepsilon * f(u) - 2\alpha(1 - \Delta)^{\alpha-1} u_\varepsilon] dx. \end{aligned}$$

By $u_\varepsilon \rightarrow u$ strongly in $L^2(\mathbf{R}^N)$ and $0 < \alpha < 1$, it is easily seen that

$$\begin{aligned} (39) \quad \lim_{\varepsilon \rightarrow 0} \int_{\mathbf{R}^N} u \rho_\varepsilon * f(u) dx &= \lim_{\varepsilon \rightarrow 0} \int_{\mathbf{R}^N} (u * \rho_\varepsilon) f(u) dx = \int_{\mathbf{R}^N} u f(u) dx = \|u\|_\alpha^2, \\ \lim_{\varepsilon \rightarrow 0} \int_{\mathbf{R}^N} (1 - \Delta)^{\alpha-1} u_\varepsilon dx &= \lim_{\varepsilon \rightarrow 0} \int_{\mathbf{R}^N} (1 + 4\pi^2|\xi|^2)^{\alpha-1} \widehat{u_\varepsilon} \overline{\widehat{u}} d\xi = \int_{\mathbf{R}^N} (1 + 4\pi^2|\xi|^2)^{\alpha-1} |\widehat{u}|^2 d\xi. \end{aligned}$$

On the other hand, recalling $u \in C_b^1$, we have

$$\begin{aligned} \int_{\mathbf{R}^N} ux \cdot \nabla(\rho_\varepsilon * f(u)) dx &= \sum_{i=1}^N \int_{\mathbf{R}^N} ux_i \partial_{x_i}(\rho_\varepsilon * f(u)) dx = - \sum_{i=1}^N \int_{\mathbf{R}^N} \partial_{x_i}(ux_i) \rho_\varepsilon * f(u) dx \\ &= -N \int_{\mathbf{R}^N} u \rho_\varepsilon * f(u) dx - \sum_{i=1}^N \int_{\mathbf{R}^N} x_i (\partial_{x_i} u) \rho_\varepsilon * f(u) dx. \end{aligned}$$

From the decay estimate of u , the same estimate holds for $\rho_\varepsilon * f(u)$ uniformly with respect to ε . Thus, letting $\varepsilon \rightarrow 0$ in the above equality, the dominated convergence theorem and (4) give us

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\mathbf{R}^N} ux \cdot \nabla(\rho_\varepsilon * f(u)) dx &= -N \int_{\mathbf{R}^N} u f(u) dx - \sum_{i=1}^N \int_{\mathbf{R}^N} x_i (\partial_{x_i} u) f(u) dx \\ (40) \quad &= -N \|u\|_\alpha^2 - \sum_{i=1}^N \int_{\mathbf{R}^N} x_i \partial_{x_i} F(u) dx \\ &= -N \|u\|_\alpha^2 + N \int_{\mathbf{R}^N} F(u) dx. \end{aligned}$$

Finally, since $\nabla u_\varepsilon(x) \rightarrow \nabla u(x)$ in $L_{\text{loc}}^\infty(\mathbf{R}^N)$ and (∇u_ε) is bounded in $L^\infty(\mathbf{R}^N)$ due to $u \in C_b^1(\mathbf{R}^N)$, one sees

$$(41) \quad \lim_{\varepsilon \rightarrow 0} \int_{\mathbf{R}^N} f(u) x \cdot \nabla u_\varepsilon dx = \int_{\mathbf{R}^N} f(u) x \cdot \nabla u dx = \int_{\mathbf{R}^N} x \cdot \nabla F(u) dx = -N \int_{\mathbf{R}^N} F(u) dx.$$

Therefore, collecting (37)–(41), we obtain

$$0 = (N - 2\alpha) \|u\|_\alpha^2 - 2N \int_{\mathbf{R}^N} F(u) dx + 2\alpha \int_{\mathbf{R}^N} (1 + 4\pi^2 |\xi|^2)^{\alpha-1} |\widehat{u}|^2 d\xi = 2P(u).$$

Thus we complete the proof. \square

Proof of Theorem 1.1. From Propositions 3.2, 3.3, 3.6 and Lemmas 2.5 and 3.4, the only task is to show that there is a positive solution u_1 of (4) which corresponds to the value c_1 .

First, select $(\gamma_n) \subset \Gamma_r$ so that $\max_{0 \leq t \leq 1} I(\gamma_n(t)) \rightarrow c_{\text{MP},r}$. As in the proof of Lemma 3.4, setting $\eta_n(t)(x) := |\gamma_n(t)|(x)$, by Lemma A.3, we observe that $I(\eta_n(t)) \leq I(\gamma_n(t))$ for every $t \in [0, 1]$ and $\eta_n \in \Gamma_r$. Hence, it follows from Lemma 3.4 that

$$\max_{0 \leq t \leq 1} \tilde{I}(0, \eta_n(t)) \rightarrow \tilde{c}_{\text{MP},r} = c_1.$$

Thus as in the proof of Proposition 3.2, applying the Ekeland's variational principle to \tilde{I} and (η_n) , and noting $\eta_n(t)(x) \geq 0$, we may find (θ_n) and $(u_n) \subset H_r^\alpha(\mathbf{R}^N)$ so that

$$I(u_n) \rightarrow c_1, \quad D_{(\theta, u)} \tilde{I}(\theta_n, u_n) \rightarrow 0, \quad \theta_n \rightarrow 0, \quad (u_n)_-(x) := \max\{0, -u(x)\} \rightarrow 0 \quad \text{in } H^\alpha(\mathbf{R}^N).$$

Repeating the argument of Proposition 3.2, $u_n \rightarrow u_0$ strongly in $H^\alpha(\mathbf{R}^N)$, $I(u_0) = c_1$, $u_0 \geq 0$, $u_0 \not\equiv 0$. Using Propositions 3.5 and 3.6, we have $u \in C_b^\beta(\mathbf{R}^N)$ for any $\beta \in (0, 2\alpha)$. Finally, a weak Harnack inequality ([21, Proposition 2]) gives $u_{1,0} > 0$ in \mathbf{R}^N and we complete the proof. \square

4. PROOF OF THEOREM 1.3

In this section, we shall prove Theorem 1.3 using Theorem 1.1. Until the proof of Theorem 1.3, we consider the more general setting. Indeed, we first assume that $f(x, s)$ in (1) satisfies the following:

(G1) $f \in C(\mathbf{R}^N \times \mathbf{R}, \mathbf{R})$ and $f(x, -s) = -f(x, s)$ for each $x \in \mathbf{R}^N$ and $s \in \mathbf{R}$.

(G2)

$$-\infty < \liminf_{s \rightarrow 0} \inf_{x \in \mathbf{R}^N} \frac{f(x, s)}{s} \leq \limsup_{s \rightarrow 0} \sup_{x \in \mathbf{R}^N} \frac{f(x, s)}{s} < 1.$$

(G3)

$$\lim_{|s| \rightarrow \infty} \sup_{x \in \mathbf{R}^N} \frac{|f(x, s)|}{|s|^{2_\alpha^*-1}} = 0.$$

(G4) There exists an $s_0 > 0$ such that

$$\inf_{x \in \mathbf{R}^N} \left(F(x, s_0) - \frac{1}{2} s_0^2 \right) > 0.$$

(G5) There exists a $f_\infty(s) \in C(\mathbf{R}, \mathbf{R})$ such that $f(x, s) \rightarrow f_\infty(s)$ in $L_{\text{loc}}^\infty(\mathbf{R})$ as $|x| \rightarrow \infty$.

Note that (G1)–(G3) and (G5) are weaker than (F1)–(F5). Moreover, we remark that (F5) implies (G4). Indeed, by the inequalities in (F5), we have

$$G(x, s) \geq \left(\frac{s}{s_1}\right)^\mu G(x, s_1) \geq c_1 s^\mu \quad \text{for all } (x, s) \in \mathbf{R}^N \times [s_1, \infty)$$

for some $c_1 > 0$. Since $\mu > 2$ and $V \in L^\infty(\mathbf{R}^N)$, we can easily find an $s_0 > 0$ such that

$$F(x, s_0) - \frac{1}{2}s_0^2 = G(x, s_0) - \frac{V(x) + 1}{2}s_0^2 \geq c_1 s_0^\mu - \frac{V(x) + 1}{2}s_0^2 > 0.$$

Thus (G4) is derived from (F5).

Under (G1)–(G5), we define the functional J corresponding to (1):

$$J(u) := \frac{1}{2}\|u\|_\alpha^2 - \int_{\mathbf{R}^N} F(x, u(x))dx \in C^1(H^\alpha(\mathbf{R}^N), \mathbf{R}).$$

Remark that a critical point of J is equivalent to a solution of (1). We begin with showing that J has the mountain pass geometry:

Proposition 4.1. *Under (G1)–(G5), there exist $\rho_0 > 0$ and $u_1 \in H^\alpha(\mathbf{R}^N)$ such that*

$$\inf_{\|u\|_\alpha = \rho_0} J(u) > 0, \quad J(u) \geq 0 \quad \text{if } \|u\|_\alpha \leq \rho_0, \quad J(u_1) < 0.$$

Proof. By (G2) and (G3), we find $\delta_0 > 0$ and $s_1 > 0$ such that

$$\sup_{x \in \mathbf{R}^N} \frac{f(x, s)}{s} \leq 1 - 2\delta_0 \quad \text{for all } |s| \leq s_1.$$

Hence,

$$\sup_{x \in \mathbf{R}^N} F(x, s) \leq \frac{1 - \delta_0}{2}s^2 \quad \text{for every } |s| \leq s_1.$$

Combining this with (G3), we have

$$\left(F(x, s) - \frac{1 - \delta_0}{2}s^2\right)_+ \leq C|s|^{2_\alpha^*} \quad \text{for each } x \in \mathbf{R}^N, s \in \mathbf{R}.$$

Hence, Sobolev's inequality yields

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|_\alpha^2 - \frac{1 - \delta_0}{2}\|u\|_{L^2}^2 - \int_{\mathbf{R}^N} \left(F(x, s) - \frac{1 - \delta_0}{2}u^2\right) dx \\ &\geq \frac{\delta_0}{2}\|u\|_\alpha^2 - \int_{\mathbf{R}^N} \left(F(x, s) - \frac{1 - \delta_0}{2}u^2\right)_+ dx \geq \frac{\delta_0}{2}\|u\|_\alpha^2 - C\|u\|_\alpha^{2_\alpha^*}. \end{aligned}$$

Choosing $\rho_0 > 0$ sufficiently small, we have

$$\inf_{\|u\|_\alpha = \rho_0} J(u) > 0, \quad J(u) \geq 0 \quad \text{if } \|u\|_\alpha \leq \rho_0.$$

For the existence of u_1 , let us consider a function defined by

$$u_1(x) := \begin{cases} s_0 & \text{if } |x| \leq R, \\ -s_0(|x| - R) + s_0 & \text{if } R < |x| \leq R + 1, \\ 0 & \text{if } |x| > R + 1 \end{cases}$$

where $s_0 > 0$ appears in (G4). Notice that $u_1 \in H^1(\mathbf{R}^N)$. By $\|u\|_\alpha \leq \|u\|_{H^1}$ for all $u \in H^1(\mathbf{R}^N)$, we obtain

$$J(u_1) \leq \frac{1}{2}\|u_1\|_{H^1}^2 - \int_{\mathbf{R}^N} F(x, u_1)dx = \frac{1}{2}\|\nabla u_1\|_{L^2}^2 - \int_{\mathbf{R}^N} F(x, u_1) - \frac{1}{2}u_1^2 dx.$$

Since it is easy to check $\|\nabla u_1\|_{L^2}^2 = O(R^{N-2})$ and $\int_{\mathbf{R}^N} F(x, u_1) - \frac{1}{2}u_1^2 dx \geq cR^N + O(R^{N-1})$ as $R \rightarrow \infty$ for some $c > 0$, for sufficiently large $R > 0$, one finds that

$$J(u_1) \leq \frac{1}{2}\|\nabla u_1\|_{L^2}^2 - \int_{\mathbf{R}^N} F(x, u_1) - \frac{1}{2}u_1^2 dx < 0.$$

Thus we complete the proof. \square

By Proposition 4.1, we define the mountain pass value of J :

$$d_{\text{MP}} := \inf_{\gamma \in \Gamma_J} \max_{0 \leq t \leq 1} J(\gamma(t)) > 0, \quad \Gamma_J := \{\gamma \in C([0, 1], H^\alpha(\mathbf{R}^N)) \mid \gamma(0) = 0, J(\gamma(1)) < 0\}.$$

As in the proof of Theorem 1.1, applying Ekeland's variational principle to J and a sequence of paths $(\gamma_n(t)) \subset \Gamma_J$ where $\gamma_n(t)(x) \geq 0$ and $\max_{0 \leq t \leq 1} J(\gamma_n(t)) \rightarrow d_{\text{MP}}$, we may find a Palais–Smale sequence (v_n) of J at level d_{MP} :

$$(42) \quad J(v_n) \rightarrow d_{\text{MP}}, \quad J(v_n) \rightarrow 0 \quad \text{strongly in } (H^\alpha(\mathbf{R}^N))^*, \quad (v_n)_- \rightarrow 0 \quad \text{strongly in } H^\alpha(\mathbf{R}^N).$$

We first observe the behaviors of bounded Palais–Smale sequences of J under (G1)–(G5). For this purpose, consider

$$(43) \quad (1 - \Delta)^\alpha \omega = f_\infty(\omega) \quad \text{in } \mathbf{R}^N, \quad \omega \in H^\alpha(\mathbf{R}^N)$$

and define the functional J_∞ corresponding to (43) by

$$J_\infty(\omega) := \frac{1}{2} \|\omega\|_\alpha^2 - \int_{\mathbf{R}^N} F_\infty(\omega) dx$$

where $F_\infty(s) := \int_0^s f_\infty(t) dt$. Notice that f_∞ satisfies (f1)–(f4) thanks to (G1)–(G5). Hence, $J_\infty \in C^1(H^\alpha(\mathbf{R}^N), \mathbf{R})$ and critical points of J are solutions of (43).

Proposition 4.2. *Suppose that (G1)–(G5) hold and that every weak solution of (43) satisfies the Pohozaev identity $P_\infty(u) = 0$ where*

$$P_\infty(u) := \frac{N - 2\alpha}{2} \|u\|_\alpha^2 - N \int_{\mathbf{R}^N} F_\infty(u) dx + \alpha \int_{\mathbf{R}^N} (1 + 4\pi^2 |\xi|^2)^{\alpha-1} |\hat{u}|^2 d\xi.$$

Let (u_n) be a bounded Palais–Smale sequence of J , that is, $(u_n) \subset H^\alpha(\mathbf{R}^N)$ is bounded and satisfies

$$J(u_n) \rightarrow c \in \mathbf{R}, \quad J'(u_n) \rightarrow 0 \quad \text{strongly in } (H^\alpha(\mathbf{R}^N))^*.$$

Then there exist $\ell \in \mathbf{N}$, $u_0 \in H^\alpha(\mathbf{R}^N)$, $\omega_i \in H^\alpha(\mathbf{R}^N)$ and $(y_{n,i})_{n=1}^\infty$ for $i = 1, \dots, \ell$ provided $\ell \geq 1$ such that

- (i) $|y_{n,i}| \rightarrow \infty$ for $1 \leq i \leq \ell$ and $|y_{n,i} - y_{n,j}| \rightarrow \infty$ for $i \neq j$ as $n \rightarrow \infty$.
- (ii) $J'(u_0) = 0$, $\omega_i \neq 0$ and $J'_\infty(\omega_i) = 0$ for $1 \leq i \leq \ell$.
- (iii) When $\ell \geq 1$,

$$\left\| u_n - u_0 - \sum_{i=1}^{\ell} \omega_i(\cdot - y_{n,i}) \right\|_\alpha \rightarrow 0, \quad c = \lim_{n \rightarrow \infty} J(u_n) = I(u_0) + \sum_{i=1}^{\ell} J_\infty(\omega_i)$$

When $\ell = 0$, $\|u_n - u_0\|_\alpha \rightarrow 0$ and $c = J(u_0)$.

We postpone a proof of Proposition 4.2 and prove it after the proof of Theorem 1.3.

Next, since f_∞ satisfies (f1)–(f4) under (G1)–(G5), we may define the mountain pass value of J_∞ :

$$d_\infty := \inf_{\gamma \in \Gamma_{J_\infty}} \max_{0 \leq t \leq 1} J_\infty(\gamma(t)) > 0, \quad \Gamma_{J_\infty} := \{\gamma \in C([0, 1], H^\alpha(\mathbf{R}^N)) \mid \gamma(0) = 0, J_\infty(\gamma(1)) < 0\}.$$

Proposition 4.3. *Let (G1)–(G5) hold. Furthermore, suppose that*

- (I) $F_\infty(s) \leq F(x, s)$ for all $(x, s) \in \mathbf{R}^N \times \mathbf{R}$.
- (II) There exists a bounded Palais–Smale sequence (u_n) of J at level d_{MP} with $(u_n)_- \rightarrow 0$ strongly in $H^\alpha(\mathbf{R}^N)$.
- (III) Every weak solution of (43) satisfies the Pohozaev identity $P_\infty(u) = 0$.

Then, (1) admits a positive solution.

Proof. By (I), we observe that $J(u) \leq J_\infty(u)$ for all $u \in H^\alpha(\mathbf{R}^N)$. Hence, $\Gamma_{J_\infty} \subset \Gamma_J$ and $0 < d_{\text{MP}} \leq d_\infty$ hold. We divide our case into two cases:

Case 1: $d_{\text{MP}} < d_\infty$ holds.

In this case, we apply Proposition 4.2 for (u_n) to obtain $u_0, \omega_1, \dots, \omega_\ell \in H^\alpha(\mathbf{R}^N)$ and $(y_{n,i})_{n=1}^\infty$ ($i = 1, \dots, \ell$) satisfying (i)–(iii) in Proposition 4.2. Remark that $J'(u_0) = 0$ and u_0 is a weak limit of (u_n) (see the proof of Proposition 4.2 below). Arguing as in the proof of Proposition 3.6, we observe that $u_0 \in C_b^\beta(\mathbf{R}^N)$ and u_0 is nonnegative. Therefore, if $u_0 \neq 0$, then a weak Harnack inequality ([21, Proposition 2]) implies that u_0 is the desired solution of (1). Thus, it suffices to show that the case $u_0 \equiv 0$ does not happen.

To this end, let $u_0 \equiv 0$ and we may assume $\ell \geq 1$ thanks to $d_{\text{MP}} > 0$. Since $\omega_i \not\equiv 0$ and $J'_\infty(\omega_i) = 0$, Theorem 1.1 and the assumption (III) assert that $J_\infty(\omega_i) \geq d_\infty > 0$ for all $1 \leq i \leq \ell$. Thus Proposition 4.2 (iii) and $u_0 \equiv 0$ yield

$$d_\infty > d_{\text{MP}} = \sum_{i=1}^{\ell} J_\infty(\omega_i) \geq \ell d_\infty.$$

Since $d_\infty > 0$, this is a contradiction. Hence, $u_0 \equiv 0$ does not occur in this case.

Case 2: $d_{\text{MP}} = d_\infty$ holds.

In this case, by Theorem 1.1, we find a positive solution $\omega \in H^\alpha(\mathbf{R}^N)$ of (43) and a path $\gamma_\omega \in \Gamma_{J_\infty}$ so that

$$(44) \quad \begin{aligned} \omega(0) &= \|\omega\|_{L^\infty}, \quad \omega \in \gamma_\omega([0, 1]), \quad \|\omega\|_{L^\infty} = \|\gamma_\omega(t)\|_{L^\infty} \text{ for } t \in (0, 1], \\ J_\infty(\omega) &= \max_{0 \leq t \leq 1} J_\infty(\gamma_\omega(t)) = d_\infty, \quad J_\infty(\gamma_\omega(t)) < J_\infty(\omega) \text{ if } \gamma_\omega(t) \neq \omega. \end{aligned}$$

Noting $\gamma_\omega(t)(\cdot - z) \in \Gamma_{J_\infty} \subset \Gamma_J$ for any $z \in \mathbf{R}^N$, let $t_z \in (0, 1)$ satisfy $\max_{0 \leq t \leq 1} J(\gamma_\omega(t)(\cdot - z)) = J(\gamma_\omega(t_z)(\cdot - z))$. Then we get

$$(45) \quad d_\infty = d_{\text{MP}} \leq J(\gamma_\omega(t_z)(\cdot - z)) \leq J_\infty(\gamma_\omega(t_z)(\cdot - z)) \leq J_\infty(\omega) = d_\infty.$$

From (44) and (45), we deduce that $\gamma_\omega(t_z) = \omega$ and

$$\int_{\mathbf{R}^N} F(x, \omega(x - z)) dx = \int_{\mathbf{R}^N} F_\infty(\omega(x - z)) dx \quad \text{for any } z \in \mathbf{R}^N.$$

Recalling $F_\infty(s) \leq F(x, s)$ for any $(x, s) \in \mathbf{R}^N \times \mathbf{R}$, we observe that

$$F(x, s) = F_\infty(s) \quad \text{for all } (x, s) \in \mathbf{R}^N \times [0, \|\omega\|_{L^\infty}].$$

This implies $f(x, s) = f_\infty(s)$ for all $x \in \mathbf{R}^N \times [0, \|\omega\|_{L^\infty}]$. From this fact, we see that ω is also a positive solution of (1). Thus we complete the proof. \square

Remark 4.4. (i) From the above proof and the existence of optimal path in Theorem 1.1, we have $d_{\text{MP}} < d_\infty$ when $F(x, s) \leq F_\infty(s)$ and $F(x, s) \not\equiv F_\infty(s)$ for each $s \in \mathbf{R}$.

(ii) In the case $\alpha = 1$, the Pohozaev identity is useful to obtain a bounded Palais–Smale sequence. For instance, we refer to [5, 28]. When $0 < \alpha < 1$, in addition to (G1)–(G4), assume that $f(x, s)$ is differentiable in x , $\nabla_x f(x, s) \in C(\mathbf{R}^N \times \mathbf{R}, \mathbf{R}^N)$ and for each $M > 0$, there exists a $C_M > 0$ such that $|\nabla_x f(x, s)| \leq C_M$ for all $(x, s) \in \mathbf{R}^N \times [-M, M]$. Under these conditions, if $u \in H^\alpha(\mathbf{R}^N) \cap C_b^1(\mathbf{R}^N)$ is a weak solution of (1), then u satisfies the following Pohozaev identity:

$$(46) \quad 0 = \frac{N - 2\alpha}{2} \|u\|_\alpha^2 - N \int_{\mathbf{R}^N} F(x, u) dx - \int_{\mathbf{R}^N} (x \cdot \nabla_x F)(x, u) dx + \alpha \int_{\mathbf{R}^N} (1 + 4\pi^2 |\xi|^2)^{\alpha-1} |\widehat{u}|^2 d\xi.$$

In fact, (46) can be proved by following the argument of Proposition 3.6 (iii) and noting

$$\sum_{i=1}^N x_i \partial_{x_i} u f(x, u) = \sum_{i=1}^N x_i \{ \partial_{x_i} (F(x, u)) - (\partial_{x_i} F(x, u)) \} = (x \cdot \nabla_x) F(x, u) - (x \cdot \nabla_x F)(x, u).$$

As in the case $\alpha = 1$, the Pohozaev identity (46) may be useful to get a bounded Palais–Smale sequence.

Now we prove Theorem 1.3.

Proof of Theorem 1.3. Let us assume (F1)–(F5). As we have already seen, (G1)–(G5) also holds. Moreover, conditions (I) and (III) in Proposition 4.3 follow from (F4) and Theorem 1.1. Thus we only need to check that (II) holds. For this purpose, we use (F5) to show that (v_n) in (42) is bounded in $H^\alpha(\mathbf{R}^N)$ and the argument is standard (for instance, see [33]). From (F5), we have

$$(47) \quad \begin{aligned} \mu d_{\text{MP}} + o(1) + o(1) \|v_n\|_\alpha &\geq \mu J(v_n) - J'(v_n) v_n \\ &= \frac{\mu - 2}{2} \left(\|v_n\|_\alpha^2 + \int_{\mathbf{R}^N} V(x) v_n^2 dx \right) - \int_{\mathbf{R}^N} \mu G(x, v_n) - g(x, v_n) v_n dx \\ &\geq \frac{\mu - 2}{2} \left(\|v_n\|_\alpha^2 + \int_{\mathbf{R}^N} V(x) v_n^2 dx \right). \end{aligned}$$

Since $\inf_{\mathbf{R}^N} V > -1$ due to (F2), a quantity defined by

$$\|u\|^2 := \|u\|_\alpha^2 + \int_{\mathbf{R}^N} V(x) u^2 dx$$

is an equivalent norm to $\|\cdot\|_\alpha$. Therefore, from (47), we infer that (v_n) is bounded in $H^\alpha(\mathbf{R}^N)$ and Proposition 4.3 implies Theorem 1.3. \square

Now we turn to prove Proposition 4.2. We first recall the following lemma due to [22, Lemma 2.1] (cf. [18, Lemma 2.18], [31, Lemma I.1] and [38, Lemma 3.1]):

Lemma 4.5. *Let $(u_n) \subset H^\alpha(\mathbf{R}^N)$ be a bounded and satisfy*

$$\sup_{y \in \mathbf{Z}^N} \int_{y+Q} |u_n|^p dx \rightarrow 0 \quad \text{for some } p \in [2, 2_\alpha^*), \quad Q := [0, 1]^N.$$

Then $u_n \rightarrow 0$ strongly in $L^q(\mathbf{R}^N)$ for all $q \in (2, 2_\alpha^)$.*

This lemma is proved in [22, Lemma 2.1], however, for the sake of readers, we show it here.

Proof. First we note that by the boundedness of (u_n) and the interpolation inequality, we may assume $2 < p$ without loss of generality. Next, we shall prove the existence of $C_0 > 0$ satisfying

$$(48) \quad \|u\|_{L^p(z+Q)} \leq C_0 \|u\|_{W^{2,\alpha}(z+Q)} \quad \text{for all } z \in \mathbf{Z}^N, \quad u \in W^{2,\alpha}(z+Q)$$

where

$$W^{2,\alpha}(\Omega) := \{u \in L^2(\Omega) \mid [u]_{W^{2,\alpha}(\Omega)} < \infty\}, \quad [u]_{W^{2,\alpha}(\Omega)}^2 := \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2\alpha}} dx dy$$

We first consider it on Q . For $u \in W^{2,\alpha}(Q)$, let

$$\tilde{u}(x', x_N) := \begin{cases} u(x', x_N) & \text{if } x_N \geq 0, \\ u(x', -x_N) & \text{if } x_N < 0. \end{cases}$$

Set $Q_1 := Q \cup (Q - \mathbf{e}_N)$ and $Rx := (x', -x_N)$ where $\mathbf{e}_N := (0, \dots, 0, 1)$. Since $|x - y| = |Rx - Ry|$ and $|x - y| \leq |Rx - y|$ for all $x, y \in Q$, and $\tilde{u}(Rx) = u(x)$ for $x \in Q$, it is easy to see that

$$\begin{aligned} \|\tilde{u}\|_{L^2(Q_1)}^2 &= 2\|u\|_{L^2(Q)}^2, \\ [\tilde{u}]_{W^{2,\alpha}(Q_1)}^2 &= \int_{Q_1 \times Q_1} \frac{|\tilde{u}(x) - \tilde{u}(y)|^2}{|x - y|^{N+2\alpha}} dx dy \\ &= \left(\int_{Q \times Q} + \int_{Q \times (Q - \mathbf{e}_N)} + \int_{(Q - \mathbf{e}_N) \times Q} + \int_{(Q - \mathbf{e}_N) \times (Q - \mathbf{e}_N)} \right) \frac{|\tilde{u}(x) - \tilde{u}(y)|^2}{|x - y|^{N+2\alpha}} dx dy \\ &\leq 4 \int_{Q \times Q} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2\alpha}} dx dy = 4[u]_{W^{2,\alpha}(Q)}^2. \end{aligned}$$

Thus $\tilde{u} \in W^{2,\alpha}(Q_1)$ and $\|\tilde{u}\|_{W^{2,\alpha}(Q_1)} \leq 2\|u\|_{W^{2,\alpha}(Q)}$.

Repeating the above argument $2N - 1$ times for edges of Q_1 except for $x_N = -1$, there exists an $Eu \in W^{2,\alpha}(Q_{2N})$ such that

$$Eu = u \quad \text{on } Q, \quad \|Eu\|_{W^{2,\alpha}(Q_{2N})} \leq 2^{2N} \|u\|_{W^{2,\alpha}(Q)}$$

where Q_{2N} is a cube satisfying $[-1, 2]^N \subset Q_{2N}$. Choosing a smooth domain $\Omega \subset \mathbf{R}^N$ such that $Q \subset \Omega \subset [-1, 2]^N$, it follows from Sobolev's embedding ([20, Theorems 5.6 and 6.7]) that

$$\|u\|_{L^p(Q)} \leq \|Eu\|_{L^p(\Omega)} \leq C_\Omega \|Eu\|_{W^{2,\alpha}(\Omega)} \leq C_\Omega \|Eu\|_{W^{2,\alpha}(Q_{2N})} \leq 2^{2N} C_\Omega \|u\|_{W^{2,\alpha}(Q)}$$

for all $u \in W^{2,\alpha}(Q)$. For (48), it is enough to translate Q , Q_{2N} and Ω . Thus (48) holds.

Now we complete the proof. We first notice from [20, Proposition 3.4] that

$$[u]_{W^{2,\alpha}(\mathbf{R}^N)}^2 = C(N, \alpha) \int_{\mathbf{R}^N} |\xi|^{2\alpha} |\hat{u}(\xi)|^2 d\xi.$$

Thus, it is easily seen that

$$\begin{aligned} \sum_{z \in \mathbf{Z}^N} [u]_{W^{2,\alpha}(z+Q)}^2 &= \sum_{z \in \mathbf{Z}^N} \int_{(z+Q) \times (z+Q)} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2\alpha}} dx dy \\ &\leq \sum_{z \in \mathbf{Z}^N} \int_{\mathbf{R}^N \times (z+Q)} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2\alpha}} dx dy = [u]_{W^{2,\alpha}(\mathbf{R}^N)}^2 \leq C \|u\|_\alpha^2. \end{aligned}$$

Therefore, by (48) and $p > 2$, we obtain

$$\begin{aligned} \|u_n\|_{L^p(\mathbf{R}^N)}^p &= \sum_{z \in \mathbf{Z}^N} \|u_n\|_{L^p(z+Q)}^p \leq \left(\sup_{z \in \mathbf{Z}^N} \|u_n\|_{L^p(z+Q)}^{p-2} \right) \sum_{z \in \mathbf{Z}^N} \|u_n\|_{L^p(z+Q)}^2 \\ &\leq \left(\sup_{z \in \mathbf{Z}^N} \|u_n\|_{L^p(z+Q)}^{p-2} \right) \sum_{z \in \mathbf{Z}^N} C_0 \|u\|_{W^{2,\alpha}(z+Q)}^2 \\ &\leq C_0 \left(\sup_{z \in \mathbf{Z}^N} \|u_n\|_{L^p(z+Q)}^{p-2} \right) \|u_n\|_\alpha^2 \rightarrow 0. \quad \square \end{aligned}$$

Now we prove Proposition 4.2.

Proof of Proposition 4.2. We argue as in [27, Proof of Proposition 4.2]. Since (u_n) is bounded in $H^\alpha(\mathbf{R}^N)$, choosing a subsequence if necessary (still denoted by (u_n)), we may assume $u_n \rightharpoonup u_0$ weakly in $H^\alpha(\mathbf{R}^N)$. From $J'(u_n) \rightarrow 0$ strongly in $(H^\alpha(\mathbf{R}^N))^*$, it is easy to check $J'(u_0) = 0$.

Next, we claim that there exist $\ell \geq 0$, $\omega_i \neq 0$ and $(y_{n,i})_{n=1}^\infty$ for $i = 1, \dots, \ell$ if $\ell \neq 0$ such that properties (i) and (ii) in Proposition 4.2 hold and

$$(49) \quad \left\| u_n - u_0 - \sum_{i=1}^{\ell} \omega_i(\cdot - y_{n,i}) \right\|_{L^p} \rightarrow 0 \quad \text{for every } p \in (2, 2_\alpha^*).$$

For this purpose, we consider

$$\limsup_{n \rightarrow \infty} \sup_{z \in \mathbf{Z}^N} \int_{z+Q} |u_n - u_0|^2 dx =: c_1.$$

If $c_1 = 0$, then Lemma 4.5 yields (49) with $\ell = 0$.

Next, consider the case $c_1 > 0$. Then we choose $(y_{n,1}) \subset \mathbf{R}^N$ such that

$$\lim_{n \rightarrow \infty} \int_{y_{n,1}+Q} |u_n - u_0|^2 dx \rightarrow c_1 > 0.$$

Let $u_n(\cdot + y_{n,1}) \rightharpoonup \omega_1$ weakly in $H^\alpha(\mathbf{R}^N)$. Since $u_n \rightarrow u_0$ strongly in $L_{\text{loc}}^2(\mathbf{R}^N)$ and $c_1 > 0$, we have $|y_{n,1}| \rightarrow \infty$ and $\omega_1 \neq 0$. Moreover, from $|y_{n,1}| \rightarrow \infty$ and $J'(u_n)[\varphi(\cdot - y_{n,1})] \rightarrow 0$ for each $\varphi \in C_0^\infty(\mathbf{R}^N)$, we also see that $J'_\infty(\omega_1) = 0$ by (G5). Since every weak solution of (43) satisfies the Pohozaev identity $P_\infty(u) = 0$ and f_∞ satisfies (f1)–(f4), by Theorem 1.1, we have $J_\infty(\omega_1) \geq d_\infty > 0$. Choosing a $\zeta_0 > 0$ so that $J_\infty(u) < d_\infty$ for all $\|u\|_\alpha < \zeta_0$, we obtain $\|\omega_1\|_\alpha \geq \zeta_0$.

Next, consider

$$\limsup_{n \rightarrow \infty} \sup_{z \in \mathbf{Z}^N} \int_{z+Q} |u_n - u_0 - \omega_1(x - y_{n,1})|^2 dx =: c_2.$$

When $c_2 = 0$, then Lemma 4.5 yields (49). On the other hand, when $c_2 > 0$, we select a $(y_{n,2}) \subset \mathbf{R}^N$ so that

$$\lim_{n \rightarrow \infty} \int_{y_{n,2}+Q} |u_n - u_0 - \omega_1(x - y_{n,1})|^2 dx = c_2.$$

Let $u_{n,2}(x + y_{n,2}) - u_0(x + y_{n,2}) - \omega_1(x - y_{n,1} + y_{n,2}) \rightharpoonup \omega_2$ weakly in $H^\alpha(\mathbf{R}^N)$. Then as in the above, it is immediate to see that

$$|y_{n,2}| \rightarrow \infty, |y_{n,1} - y_{n,2}| \rightarrow \infty, u_n(x + y_{n,2}) \rightharpoonup \omega_2 \neq 0, J'_\infty(\omega_2) = 0, J_\infty(\omega_2) \geq d_\infty, \|\omega_2\|_\alpha \geq \zeta_0.$$

Now we repeat the same procedure. Namely, consider

$$\limsup_{n \rightarrow \infty} \sup_{z \in \mathbf{Z}^N} \int_{z+Q} |u_n - u_0 - \omega_1(x - y_{n,1}) - \omega_2(x - y_{n,2})|^2 dx =: c_3$$

and find $\omega_3 \neq 0$ and $(y_{n,3})$ if $c_3 > 0$. Therefore, we obtain $\ell \in \mathbf{N}$, $\omega_i \neq 0$ ($1 \leq i \leq \ell$) and $(y_{n,i})_{n=1}^\infty$ ($1 \leq i \leq \ell$) so that

$$(50) \quad \begin{aligned} &|y_{n,i}| \rightarrow \infty, |y_{n,i} - y_{n,j}| \rightarrow \infty \text{ if } i \neq j, u_n(x + y_{n,i}) \rightharpoonup \omega_i \neq 0, \\ &J'_\infty(\omega_i) = 0, J_\infty(\omega_i) \geq d_\infty, \|\omega_i\|_\alpha \geq \zeta_0. \end{aligned}$$

To prove (49), it suffices to prove that this procedure cannot be iterated infinitely many times and

$$(51) \quad \limsup_{n \rightarrow \infty} \sup_{z \in \mathbf{Z}^N} \int_{z+Q} \left| u_n - u_0 - \sum_{i=1}^{\ell} \omega_i(x - y_{n,i}) \right|^2 dx = 0 \quad \text{for some } \ell \in \mathbf{N}.$$

To see this, from (50) it follows that

$$\begin{aligned}
0 &\leq \left\| u_n - u_0 - \sum_{i=1}^{\ell} \omega_i(\cdot - y_{n,i}) \right\|_{\alpha}^2 = \|u_n\|_{\alpha}^2 + \|u_0\|_{\alpha}^2 + \sum_{i=1}^{\ell} \|\omega_i\|_{\alpha}^2 \\
&\quad - 2 \langle u_n, u_0 \rangle_{\alpha} - 2 \sum_{i=1}^{\ell} \langle u_n, \omega_i(\cdot - y_{n,i}) \rangle_{\alpha} \\
&\quad + 2 \sum_{i=1}^{\ell} \langle u_0, \omega_i(\cdot - y_{n,i}) \rangle_{\alpha} + 2 \sum_{1 \leq i < j \leq \ell} \langle \omega_i(\cdot - y_{n,i}), \omega_j(\cdot - y_{n,j}) \rangle_{\alpha} \\
&= \|u_n\|_{\alpha}^2 - \|u_0\|_{\alpha}^2 - \sum_{i=1}^{\ell} \|\omega_i\|_{\alpha}^2 + o(1) \\
&\leq \|u_n\|_{\alpha}^2 - \|u_0\|_{\alpha}^2 - \ell \zeta_0 + o(1).
\end{aligned}$$

Since (u_n) is bounded, we observe that the above procedure cannot be iterated infinitely many times. Therefore, (51) holds, which implies (49).

Finally, we shall prove that

$$(52) \quad \left\| u_n - u_0 - \sum_{i=1}^{\ell} \omega_i(\cdot - y_{n,i}) \right\|_{\alpha} \rightarrow 0.$$

To do this, set $U_n(x) := u_n(x) - u_0(x) - \sum_{i=1}^{\ell} \omega_i(x - y_{n,i})$. By (G2), select $\delta_0 > 0$ and $s_1 > 0$ so that

$$(53) \quad f(x, s)s \leq (1 - \delta_0)s^2 \quad \text{for all } (x, s) \in \mathbf{R}^N \times [-s_1, s_1].$$

It is clear that a norm

$$\|u\|^2 := \|u\|_{\alpha}^2 - (1 - \delta_0)\|u\|_{L^2}^2$$

is equivalent to $\|\cdot\|_{\alpha}$. Therefore, instead of (52), we shall show $\|U_n\| \rightarrow 0$.

To this end, putting $f_{\delta_0}(x, s) := f(x, s) - (1 - \delta_0)s$ and $f_{\infty, \delta_0}(s) := f_{\infty}(s) - (1 - \delta_0)s$, we first notice from $J'(u_n) \rightarrow 0$, $J'(u_0) = 0$ and $J'_{\infty}(\omega_i) = 0$ that

$$\begin{aligned}
\|U_n\|^2 &= \|U_n\|_{\alpha}^2 - (1 - \delta_0)\|U_n\|_{L^2}^2 \\
&= \left\langle u_n - u_0 - \sum_{i=1}^{\ell} \omega_i(\cdot - y_{n,i}), U_n \right\rangle_{\alpha} - (1 - \delta_0) \left\langle u_n - u_0 - \sum_{i=1}^{\ell} \omega_i(\cdot - y_{n,i}), U_n \right\rangle_{L^2} \\
&= \int_{\mathbf{R}^N} f_{\delta_0}(x, u_n) U_n dx - \int_{\mathbf{R}^N} f_{\delta_0}(x, u_0) U_n dx - \sum_{i=1}^{\ell} \int_{\mathbf{R}^N} f_{\infty, \delta_0}(\omega_i(x - y_{n,i})) U_n dx + o(1) \\
(54) \quad &= \int_{\mathbf{R}^N} \left\{ f_{\delta_0}(x, u_n) - f_{\delta_0}(x, u_0) - \sum_{i=1}^{\ell} f_{\delta_0}(x, \omega_i(x - y_{n,i})) \right\} U_n dx \\
&\quad + \sum_{i=1}^{\ell} \int_{\mathbf{R}^N} \{f(x, \omega_i(x - y_{n,i})) - f_{\infty}(\omega_i(x - y_{n,i}))\} U_n dx + o(1) \\
&=: I_n + II_n + o(1).
\end{aligned}$$

We shall show $I_n = o(1) = II_n$. We first consider I_n . For any $M > 0$, by Hölder's inequality, we have

$$\begin{aligned}
&\int_{\{|U_n| \geq M\}} \left| f_{\delta_0}(x, u_n) - f_{\delta_0}(x, u_0) - \sum_{i=1}^{\ell} f_{\delta_0}(x, \omega_i(x - y_{n,i})) \right| |U_n| dx \\
(55) \quad &\leq \|U_n\|_{L^{2^*}_{\alpha}(\{|U_n| \geq M\})} \left(\|f_{\delta_0}(x, u_n)\|_{L^{p^*}(\{|U_n| \geq M\})} + \|f_{\delta_0}(x, u_0)\|_{L^{p^*}(\{|U_n| \geq M\})} \right. \\
&\quad \left. + \sum_{i=1}^{\ell} \|f_{\delta_0}(x, \omega_i(x - y_{n,i}))\|_{L^{p^*}(\{|U_n| \geq M\})} \right)
\end{aligned}$$

where $p^* := 2^*_{\alpha}/(2^*_{\alpha} - 1)$. Since (U_n) is bounded in $L^{2^*_{\alpha}}(\mathbf{R}^N)$ and $p^* < 2$, we have

$$C_1 \geq \|U_n\|_{L^{2^*_{\alpha}}(\{|U_n| \geq M\})}^{2^*_{\alpha}} \geq M^{2^*_{\alpha}} \mathcal{L}^N(\{|U_n| \geq M\}), \quad \|u\|_{L^{p^*}(\{|U_n| \geq M\})}^{p^*} \leq \mathcal{L}^N(\{|U_n| \geq M\})^{1-p^*/2} \|u\|_{L^2}^{p^*}$$

where $C_1 > 0$ is independent of n . In particular, $\sup_{n \geq 1} \mathcal{L}^N(|U_n| \geq M) \rightarrow 0$ as $M \rightarrow \infty$. Recalling $|f_{\delta_0}(x, s)| \leq c_\varepsilon |s| + \varepsilon |s|^{2_\alpha^*}$ for all $(x, s) \in \mathbf{R}^N \times \mathbf{R}$, it follows from Hölder's inequality and the boundedness of (u_n) that

$$\begin{aligned} & \sup_{n \geq 1} \left\{ \|f_{\delta_0}(x, u_n)\|_{L^{p^*}(|U_n| \geq M)}^{p^*} + \|f_{\delta_0}(x, u_0)\|_{L^{p^*}(|U_n| \geq M)}^{p^*} + \|f_{\delta_0}(x, \omega_i(x - y_{i,n}))\|_{L^{p^*}(|U_n| \geq M)}^{p^*} \right\} \\ & \leq \sup_{n \geq 1} \int_{|U_n| \geq M} \left\{ c_\varepsilon |u_n|^{p^*} + \varepsilon |u_n|^{2_\alpha^*} + c_\varepsilon |u_0|^{p^*} + \varepsilon |u_0|^{2_\alpha^*} \right. \\ & \quad \left. + \sum_{i=1}^\ell \left(c_\varepsilon |\omega_i(x - y_{n,i})|^{p^*} + \varepsilon |\omega_i(x - y_{n,i})|^{2_\alpha^*} \right) \right\} dx \\ & \leq c_\varepsilon \mathcal{L}^N(|U_n| \geq M)^{1-p^*/2} + C_2 \varepsilon \end{aligned}$$

for some $C_2 > 0$. Therefore, by (55) and $\sup_{n \geq 1} \mathcal{L}^N(|U_n| \geq M) \rightarrow 0$ as $M \rightarrow \infty$, we get

$$\limsup_{M \rightarrow \infty} \sup_{n \geq 1} \int_{|U_n| \geq M} \left| f_{\delta_0}(x, u_n) - f_{\delta_0}(x, u_0) - \sum_{i=1}^\ell f_{\delta_0}(x, \omega_i(x - y_{i,n})) \right| |U_n| dx \leq C_2 \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we deduce that

$$(56) \quad \limsup_{M \rightarrow \infty} \sup_{n \geq 1} \int_{|U_n| \geq M} \left| f_{\delta_0}(x, u_n) - f_{\delta_0}(x, u_0) - \sum_{i=1}^\ell f_{\delta_0}(x, \omega_i(x - y_{i,n})) \right| |U_n| dx = 0.$$

On the other hand, denote by $\chi_n^M(x) := \chi_{|U_n| \leq M}(x)$ the characteristic function of $|U_n| \leq M$. Since $u_n \rightarrow u_0$ and $u_n(x + y_{n,i}) \rightarrow \omega_i$ strongly in $L_{\text{loc}}^p(\mathbf{R}^N)$ for every $p < 2_\alpha^*$, for all $R > 0$, Strauss' lemma, (G2), (G3), (G5) and the facts $|y_{n,i}| \rightarrow \infty$ and $|y_{n,i} - y_{n,j}| \rightarrow \infty$ for $i \neq j$ yield

$$\begin{aligned} & \int_{B_R(0)} \chi_n^M(x) \left\{ |f_{\delta_0}(x, u_n) - f_{\delta_0}(x, u_0)| + \sum_{i=1}^\ell |f_{\delta_0}(x, \omega_i(x - y_{n,i}))| \right\} |U_n| dx \\ & \leq M \int_{B_R(0)} |f_{\delta_0}(x, u_n) - f_{\delta_0}(x, u_0)| + \sum_{i=1}^\ell |f_{\delta_0}(x, \omega_i(x - y_{n,i}))| dx \rightarrow 0 \quad \text{as } n \rightarrow \infty, \\ (57) \quad & \int_{B_R(y_{n,i})} \chi_n^M(x) \left\{ |f_{\delta_0}(x, u_0)| + \sum_{j \neq i} |f_{\delta_0}(x, \omega_j(x - y_{n,j}))| \right\} |U_n| dx \\ & \quad + \int_{B_R(y_{n,i})} \chi_n^M(x) |f_{\delta_0}(x, u_n) - f_{\delta_0}(x, \omega_i(x - y_{n,i}))| |U_n| dx \\ & \leq M \int_{B_R(0)} \left\{ |f_{\delta_0}(x + y_{n,i}, u_0(x + y_{n,i}))| + \sum_{j \neq i} |f_{\delta_0}(x + y_{n,i}, \omega_j(x + y_{n,i} - y_{n,j}))| \right. \\ & \quad \left. + |f_{\delta_0}(x + y_{n,i}, u_n(x + y_{n,i})) - f_{\delta_0}(x + y_{n,i}, \omega_i(x))| \right\} dx \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Writing $V_R := \mathbf{R}^N \setminus (B_R(0) \cup \bigcup_{i=1}^\ell B_R(y_{n,i}))$ and recalling $|f_{\delta_0}(x, s)| \leq c_1(|s| + |s|^{2_\alpha^*})$, we have

$$(58) \quad \begin{aligned} \int_{V_R} \chi_n^M |f_{\delta_0}(x, u_0) U_n| dx & \leq c_1 \int_{V_R} (|u_0| + |u_0|^{2_\alpha^*-1}) |U_n| dx \\ & \leq c_1 \left(\|u_0\|_{L^2(V_R)} \|U_n\|_{L^2(V_R)} + \|u_0\|_{L^{2_\alpha^*}(V_R)}^{2_\alpha^*-1} \|U_n\|_{L^{2_\alpha^*}} \right) = o_R(1) \end{aligned}$$

where $o_R(1) \rightarrow 0$ as $R \rightarrow 0$ uniformly in n and $M \geq 1$. Similarly, we also obtain

$$(59) \quad \int_{V_R} \chi_n^M |f_{\delta_0}(x, \omega_i(x - y_{n,i}))| |U_n| dx + \int_{V_R} \chi_n^M |f_{\delta_0}(x, u_n)| \left(|u_0| + \sum_{i=1}^\ell |\omega_i(x - y_{n,i})| \right) dx = o_R(1).$$

Finally, noting that $f_{\delta_0}(x, s)s \leq 0$ for all $|s| \leq s_1$ and that $|f_{\delta_0}(x, s)s| \leq \varepsilon |s|^{2_\alpha^*} + c_\varepsilon |s|^{p_0}$ for all $x \in \mathbf{R}^N$ and $|s| \geq s_1$ where $p_0 \in (2, 2_\alpha^*)$, by (53), we have

$$\begin{aligned} \int_{V_R} f_{\delta_0}(x, u_n) \chi_n^M u_n dx & = \int_{V_R} f_{\delta_0}(x, \chi_n^M u_n) \chi_n^M u_n dx \leq \int_{V_R \cap \{|\chi_n^M u_n| \geq s_1\}} f_{\delta_0}(x, \chi_n^M u_n) \chi_n^M u_n dx \\ & \leq \varepsilon \|\chi_n^M u_n\|_{L^{2_\alpha^*}(V_R)}^{2_\alpha^*} + c_\varepsilon \|\chi_n^M u_n\|_{L^{p_0}(V_R)}^{p_0}. \end{aligned}$$

Recalling (49), $2 < p_0 < 2_\alpha^*$ and the definition of V_R , we obtain

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \|\chi_n^M u_n\|_{L^{p_0}(V_R)} \\ & \leq \limsup_{n \rightarrow \infty} \left(\left\| u_n - u_0 - \sum_{i=1}^{\ell} \omega_i(\cdot - y_{n,i}) \right\|_{L^{p_0}(V_R)} + \|u_0\|_{L^{p_0}(V_R)} + \sum_{i=1}^{\ell} \|\omega_i(\cdot - y_{n,i})\|_{L^{p_0}(V_R)} \right) \\ & = o_R(1), \end{aligned}$$

which implies $\limsup_{n \rightarrow \infty} \left| \int_{V_R} f_{\delta_0}(x, u_n) \chi_n^M u_n dx \right| \leq c\varepsilon + c_\varepsilon o_R(1)$ for some $c > 0$. Thus,

$$(60) \quad \limsup_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \left| \int_{V_R} f_{\delta_0}(x, u_n) \chi_n^M u_n dx \right| \leq c\varepsilon.$$

Collecting (58), (59) and (60) with $U_n = u_n - u_0 - \sum_{i=1}^{\ell} \omega_i(x - y_{n,i})$, we observe that

$$(61) \quad \limsup_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \left| \int_{V_R} \left\{ f_{\delta_0}(x, u_n) - f_{\delta_0}(x, u_0) - \sum_{i=1}^{\ell} f_{\delta_0}(x, \omega_i(x - y_{n,i})) \right\} \chi_n^M U_n dx \right| \leq c\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, by (57) and (61), we obtain

$$\limsup_{n \rightarrow \infty} \left| \int_{\mathbf{R}^N} \left\{ f_{\delta_0}(x, u_n) - f_{\delta_0}(x, u_0) - \sum_{i=1}^{\ell} f_{\delta_0}(x, \omega_i(x - y_{n,i})) \right\} \chi_n^M U_n dx \right| = 0.$$

Combining this with (56), we observe that

$$\limsup_{n \rightarrow \infty} |I_n| = 0.$$

In a similar way, we can also prove that $\limsup_{n \rightarrow \infty} |II_n| = 0$. Hence, by (54), we get $\|U_n\| \rightarrow 0$ as $n \rightarrow \infty$ and this completes the proof. \square

APPENDIX A. SOME TECHNICAL LEMMAS

Here we prove some technical results. First, we show the following:

Proposition A.1. (i) Let $0 < \delta_0 < 1$ and define $m(\xi)$ by

$$m(\xi) := \frac{1}{(1 + 4\pi^2|\xi|^2)^\alpha - (1 - \delta_0)}.$$

Set $K(x) := \mathcal{F}^{-1}m$. Then $K(x) \in C^\infty(\mathbf{R}^N \setminus \{0\})$ and for any $k > 0$ there exists a $c_k > 0$ such that $|K(x)| \leq c_k(\chi_{B_1(0)}(x)|x|^{-N+2\alpha} + \chi_{B_1(0)^c}(x)|x|^{-k})$ for all $x \in \mathbf{R}^N$.

(ii) For any $g \in L^2(\mathbf{R}^N)$, the equation

$$(62) \quad (1 - \Delta)^\alpha v - (1 - \delta_0)v = g(x) \quad \text{in } \mathbf{R}^N$$

has a unique solution $v \in H^\alpha(\mathbf{R}^N)$. Moreover, when $g \in L^2(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$ and $g(x) \geq 0, \neq 0$, then $v \in C_b^\beta(\mathbf{R}^N)$ for any $0 < \beta < 2\alpha$ and $v > 0$ in \mathbf{R}^N . In addition, if $\text{supp } g$ is compact and $g \in L^\infty(\mathbf{R}^N)$ with $g(x) \geq 0$, then for any $k \in \mathbf{R}^N$ one finds a $c_k > 0$ such that $v(x) \leq c_k(1 + |x|)^{-k}$ for all $x \in \mathbf{R}^N$.

(iii) Suppose that $g_i \in L^2(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$ ($i = 1, 2$) satisfy $g_1 \leq g_2$. Let $v_i \in H^\alpha(\mathbf{R}^N)$ be a unique solution of (62) with $g(x) = g_i(x)$. Then $v_1(x) \leq v_2(x)$ for $x \in \mathbf{R}^N$.

Proof. (i) The smoothness of K and the inequality $K(x) \leq C_0|x|^{-N+2\alpha}$ for $|x| \leq 1$ follow from the arguments in [36, §4.4 of Chapter VI]. For the decay estimate at infinity, since $(\Delta_\xi)^k m(\xi) \in L^1(\mathbf{R}^N)$ provided $k > N/2$, we have $(4\pi^2|x|^2)^k K(x) = \mathcal{F}^{-1}(\Delta^k m) \in L^\infty(\mathbf{R}^N)$. From this, the desired estimate follows.

(ii) First, notice that a norm defined by

$$\|u\|^2 := (u, u), \quad (u, v) := \langle u, v \rangle_\alpha - (1 - \delta_0) \langle u, v \rangle_{L^2} \quad \text{for } u, v \in H^\alpha(\mathbf{R}^N)$$

is equivalent to $\|\cdot\|_\alpha$ since $0 < \delta_0 < 1$. Hence, (62) has a unique solution $v \in H^\alpha(\mathbf{R}^N)$ for any $g \in L^2(\mathbf{R}^N)$ due to the Lax-Milgram Theorem and it is expressed as $v = K * g$. Since (62) is rewritten as $(1 - \Delta)^\alpha v = (1 - \delta_0)v + g$, we obtain

$$v = G_{2\alpha} * ((1 - \delta_0)v + g).$$

Thus if $g \in L^\infty(\mathbf{R}^N) \cap L^2(\mathbf{R}^N)$, using the bootstrap argument and $\mathcal{L}_{2\alpha}^p \subset W^{2\alpha, p}(\mathbf{R}^N)$, one can check $v \in C_b^{2\beta}(\mathbf{R}^N)$ for all $\beta \in (0, 2\alpha)$.

Let us assume $g(x) \geq 0, \neq 0$ and $g \in L^\infty(\mathbf{R}^N)$. Since $v = G_{2\alpha} * ((1 - \delta_0)v + g)$, $G_{2\alpha} > 0$ and $(G_{2\alpha} * g) > 0$, we observe that

$$v(x) = \int_{\mathbf{R}^N} G_{2\alpha}(y) \{(1 - \delta_0)v(x - y) + g(x - y)\} dy > (1 - \delta_0) \int_{\mathbf{R}^N} G_{2\alpha}(y)v(x - y) dy.$$

Noting $v \in C_b^\beta(\mathbf{R}^N)$ and $v(x) \rightarrow 0$ as $|x| \rightarrow \infty$, if $v(x_0) = \min_{\mathbf{R}^N} v$ holds for some $x_0 \in \mathbf{R}^N$, then it follows from $\|G_{2\alpha}\|_{L^1} = 1$ that

$$v(x_0) > (1 - \delta_0) \int_{\mathbf{R}^N} G_{2\alpha}(y)v(x_0 - y) dy \geq (1 - \delta_0) \int_{\mathbf{R}^N} G_{2\alpha}(y)v(x_0) dy = (1 - \delta_0)v(x_0).$$

By $0 < \delta_0 < 1$, we get $\min_{\mathbf{R}^N} v = v(x_0) > 0$, however, this contradicts $v(x) \rightarrow 0$ as $|x| \rightarrow \infty$. Hence, v does not have any global minimum on \mathbf{R}^N and this asserts $v(x) > 0$ for each $x \in \mathbf{R}^N$.

Finally, when $g(x)$ has the compact support, since $v = K * g$ and K decays faster than any polynomial thanks to (i), it is easily seen that $v(x)$ also decays faster than any polynomial.

(iii) Set $w := v_2 - v_1$. We observe that w satisfies

$$(1 - \Delta)^\alpha w - (1 - \delta_0)w = g_2 - g_1 \geq 0 \quad \text{in } \mathbf{R}^N.$$

Using (ii), one has $w \geq 0$ in \mathbf{R}^N and $v_2(x) \geq v_1(x)$ in \mathbf{R}^N . □

Next, in order to prove Proposition 3.5 and (34), we consider the extension problem observed in [21] (cf. [10]). For $X = (x, t) \in \mathbf{R}_+^{N+1} := \mathbf{R}^N \times (0, \infty)$ and $u \in H^\alpha(\mathbf{R}^N)$, consider

$$(63) \quad \begin{cases} t^{1-2\alpha}(-\Delta_x + 1)w - (t^{1-2\alpha}w_t)_t = 0 & \text{in } \mathbf{R}_+^{N+1}, \\ w = u & \text{on } \mathbf{R}^N \end{cases}$$

where $\Delta_x = \sum_{i=1}^N \partial_{x_i}^2$. We set

$$X^\alpha := \{w(x, t) : \mathbf{R}_+^{N+1} \rightarrow \mathbf{R} \mid \|w\|_{X^\alpha} < \infty\}, \quad \|w\|_{X^\alpha}^2 := \int_{\mathbf{R}_+^{N+1}} t^{1-2\alpha} (|\nabla w|^2 + w^2) dX$$

where $\nabla = (\nabla_x, \partial_t)$. First we collect some facts. See, for instance, [19, 21].

Proposition A.2. (i) *There exists the trace operator $\text{Tr} : X^\alpha \rightarrow H^\alpha(\mathbf{R}^N)$.*

(ii) *For any $u \in H^\alpha(\mathbf{R}^N)$, (63) has a unique solution $w = Eu \in X^\alpha$. Furthermore, there exists a $\kappa_\alpha > 0$ such that Eu satisfies*

$$\int_{\mathbf{R}_+^{N+1}} t^{1-2\alpha} (\nabla Eu \cdot \nabla \varphi + Eu \varphi) dX = \kappa_\alpha \langle u, \text{Tr } \varphi \rangle_\alpha$$

for all $u \in H^\alpha(\mathbf{R}^N)$ and $\varphi \in X^\alpha$.

(iii) *For every $u \in H^\alpha(\mathbf{R}^N)$ and $w \in X^\alpha$ with $\text{Tr } w = u$, one has*

$$\kappa_\alpha \|u\|_\alpha^2 = \|Eu\|_{X^\alpha}^2 \leq \|w\|_{X^\alpha}^2.$$

(iv) *If $u \in H^\alpha(\mathbf{R}^N)$ with $u \geq 0$, then $Eu \geq 0$ in \mathbf{R}_+^{N+1} .*

Using these properties, we first show (34), namely,

Lemma A.3. *For any $u \in H^\alpha(\mathbf{R}^N)$, $\| |u| \|_\alpha \leq \|u\|_\alpha$. Moreover, the map $u \mapsto |u| : H^\alpha(\mathbf{R}^N) \rightarrow H^\alpha(\mathbf{R}^N)$ is continuous.*

Proof. Let $u \in H^\alpha(\mathbf{R}^N)$. Then it is easily seen that $\| |Eu| \|_\alpha = \|Eu\|_\alpha < \infty$, hence, $|Eu| \in X^\alpha$. We can also check that $\text{Tr } |Eu| = |u|$. Thus, by Proposition A.2 (iii), we have $\kappa_\alpha \| |u| \|_\alpha^2 \leq \| |Eu| \|_{X^\alpha}^2 = \|Eu\|_{X^\alpha}^2 = \kappa_\alpha \|u\|_\alpha^2$. For the continuity of the map $u \mapsto |u|$, let $u_n \rightarrow u_0$ in $H^\alpha(\mathbf{R}^N)$. From $Eu_n \rightarrow Eu_0$ in X^α due to Proposition A.2 (iii), we have $|Eu_n| \rightarrow |Eu_0|$ in X^α . By $\text{Tr } |Eu_n| = |u_n|$ and the boundedness of Tr , we have $|u_n| = \text{Tr } |Eu_n| \rightarrow \text{Tr } |Eu_0| = |u_0|$ in $H^\alpha(\mathbf{R}^N)$. □

Now we prove Proposition 3.5.

Proof of Proposition 3.5. The argument is similar to the case $\alpha = 1$ (see [9]). For $k \in \mathbf{N}$, set

$$a_k(x) := k \quad \text{if } a(x) \geq k, \quad := a(x) \quad \text{if } |a(x)| < k, \quad := -k \quad \text{if } a(x) \leq -k.$$

Then thanks to (36) and $A \in L^{N/(2\alpha)}(\mathbf{R}^N)$, we have

$$(64) \quad |a(x) - a_k(x)| \leq C_0 A(x) \quad \text{for each } (x, k) \in \mathbf{R}^N \times [C_0, \infty), \quad \|a - a_k\|_{L^{N/(2\alpha)}(\mathbf{R}^N)} \rightarrow 0.$$

The first step is to show:

Step 1: For any $\varepsilon > 0$ there exists a $\lambda_\varepsilon > 0$ such that

$$\int_{\mathbf{R}^N} |a|v^2 dx + \int_{\mathbf{R}^N} |a_k|v^2 dx \leq \varepsilon[v]_{H^\alpha}^2 + \lambda_\varepsilon \|v\|_{L^2}^2 \quad \text{for all } v \in H^\alpha(\mathbf{R}^N), \quad k \geq 1$$

where $[u]_{H^\alpha}^2 := \int_{\mathbf{R}^N \times \mathbf{R}^N} |u(x) - u(y)|^2 / |x - y|^{N+2\alpha} dx dy$.

From (36) and the definition of a_k , it follows that $|a(x)| + |a_k(x)| \leq C_0(1 + A(x))$. Therefore, it suffices to prove

$$(65) \quad \int_{\mathbf{R}^N} A v^2 dx \leq \varepsilon[v]_{H^\alpha}^2 + \lambda_\varepsilon \|v\|_{L^2}^2.$$

We first notice that for $n \geq 1$,

$$\int_{\mathbf{R}^N} A(x)v^2 dx = \int_{[A < n]} A(x)v^2 dx + \int_{[A \geq n]} A(x)v^2 dx \leq n\|v\|_{L^2}^2 + \int_{[A \geq n]} A(x)v^2 dx.$$

Using Hölder's inequality and Sobolev's inequality for the second term, we obtain

$$\int_{[A \geq n]} A(x)v^2 dx \leq \|A\|_{L^{N/(2\alpha)}([A \geq n])} \|v\|_{L^{2_\alpha^*}}^2 \leq C_S \|A\|_{L^{N/(2\alpha)}([A \geq n])} [v]_{H^\alpha}^2.$$

Thus

$$\int_{\mathbf{R}^N} A(x)v^2 dx \leq n\|v\|_{L^2}^2 + C_S \|A\|_{L^{N/(2\alpha)}([A \geq n])} [v]_{H^\alpha}^2.$$

Since $A \in L^{N/(2\alpha)}(\mathbf{R}^N)$, choosing n large enough, we get (65) and Step 1 holds.

Step 2: The operators $(1 - \Delta)^\alpha - a(x) + \lambda_\varepsilon$ and $(1 - \Delta)^\alpha - a_k(x) + \lambda_\varepsilon$ are coercive on $H^\alpha(\mathbf{R}^N)$ for all sufficiently small $\varepsilon > 0$.

By Step 1, for sufficiently small $\varepsilon > 0$, one sees that

$$(66) \quad \begin{aligned} & \int_{\mathbf{R}^N} (1 + 4\pi^2|\xi|^2)^\alpha |\widehat{v}(\xi)|^2 d\xi - \int_{\mathbf{R}^N} a(x)v^2 dx + \lambda_\varepsilon \|v\|_{L^2}^2 \\ & \geq \int_{\mathbf{R}^N} (1 + 4\pi^2|\xi|^2)^\alpha |\widehat{v}(\xi)|^2 d\xi - \varepsilon[v]_{H^\alpha}^2 \geq \frac{1}{2} \int_{\mathbf{R}^N} (1 + 4\pi^2|\xi|^2)^\alpha |\widehat{v}(\xi)|^2 d\xi \end{aligned}$$

and

$$(67) \quad \int_{\mathbf{R}^N} (1 + 4\pi^2|\xi|^2)^\alpha |\widehat{v}(\xi)|^2 d\xi - \int_{\mathbf{R}^N} a_k(x)v^2 dx + \lambda_\varepsilon \|v\|_{L^2}^2 \geq \frac{1}{2} \int_{\mathbf{R}^N} (4\pi^2|\xi|^2 + m^2)^\alpha |\widehat{v}(\xi)|^2 d\xi.$$

Hence, Step 2 holds.

Rewrite (35) as follows:

$$(68) \quad (1 - \Delta)^\alpha u - a(x)u + \lambda_\varepsilon u = \lambda_\varepsilon u \quad \text{in } \mathbf{R}^N.$$

Noting Step 2, we may find a unique solution $\psi_k \in H^\alpha(\mathbf{R}^N)$ of

$$(1 - \Delta)^\alpha \psi_k - a_k(x)\psi_k + \lambda_\varepsilon \psi_k = \lambda_\varepsilon u \quad \text{in } \mathbf{R}^N$$

for sufficiently small $\varepsilon > 0$. From (67), one observes that (ψ_k) is bounded in $H^\alpha(\mathbf{R}^N)$. Furthermore, since u is a unique solution of (68) thanks to (66), we also see from (64) that

$$(69) \quad \psi_k \rightarrow u \quad \text{strongly in } H^\alpha(\mathbf{R}^N).$$

Next, let $w_k \in X^\alpha$ be a unique solution of (63) with $u = \psi_k$. For $n \in \mathbf{N}$, set

$$\psi_{k,n}(x) := \begin{cases} n & \text{if } \psi_k(x) \geq n, \\ \psi_k(x) & \text{if } |\psi_k(x)| < n, \\ -n & \text{if } \psi_k(x) \leq -n, \end{cases} \quad w_{k,n}(X) := \begin{cases} n & \text{if } w_k(X) \geq n, \\ w_k(X) & \text{if } |w_k(X)| < n, \\ -n & \text{if } w_k(X) \leq -n, \end{cases}$$

Remark that for every $p \geq 2$,

$$\begin{aligned} |w_{k,n}|^{p-2} w_{k,n} &\in X^\alpha \cap L^\infty(\mathbf{R}_+^{N+1}), \quad |\psi_{k,n}|^{p-2} \psi_{k,n} \in H^\alpha(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N), \\ \text{Tr}(|w_{k,n}|^{p-2} w_{k,n}) &= |\psi_{k,n}|^{p-2} \psi_{k,n}, \quad \|w_{k,n} - w_k\|_{X^\alpha} \rightarrow 0, \quad \|\psi_{k,n} - \psi_k\|_\alpha \rightarrow 0. \end{aligned}$$

Step 3: Assume that $u, \psi_k \in L^p(\mathbf{R}^N)$ and $\psi_k \rightarrow u$ strongly in $L^p(\mathbf{R}^N)$ for some $p > 2$. Then

$$|u|^{p/2} \in H^\alpha(\mathbf{R}^N), \quad \kappa_\alpha \| |u|^{p/2} \|_{H^\alpha}^2 \leq \| |w|^{p/2} \|_{X^\alpha}^2 \leq C_1 \|u\|_{L^p}^p \quad \text{where } w = Eu \in X^\alpha.$$

We use $|w_{k,n}|^{p-2}w_{k,n}$ as a test function to (63) with $w = w_k$ to get

$$\begin{aligned} & \int_{\mathbf{R}_+^{N+1}} t^{1-2\alpha} \{ \nabla w_k \cdot \nabla (|w_{k,n}|^{p-2}w_{k,n}) + w_k |w_{k,n}|^{p-2}w_{k,n} \} dX \\ &= \kappa_\alpha \int_{\mathbf{R}^N} (a_k \psi_k - \lambda_\varepsilon \psi_k + \lambda_\varepsilon u) |\psi_{k,n}|^{p-2} \psi_{k,n} dx. \end{aligned}$$

Notice that

$$\nabla w_k \cdot \nabla (|w_{k,n}|^{p-2}w_{k,n}) = (p-1) |w_{k,n}|^{p-2} |\nabla w_{k,n}|^2 = \frac{4}{p^2} (p-1) |\nabla (|w_{k,n}|^{p/2})|^2.$$

Furthermore, by $|w_{k,n}| \leq |w_k|$, $|\psi_{k,n}| \leq |\psi_k|$, $w_k w_{k,n} = |w_k| |w_{k,n}|$, $\psi_{k,n} \psi_k = |\psi_{k,n}| |\psi_k|$, $|a_k(x)| \leq k$ and $|\psi_{k,n}|^{p/2} \in H^\alpha(\mathbf{R}^N)$, it follows from Step 1 that

$$\begin{aligned} \int_{\mathbf{R}_+^{N+1}} t^{1-2\alpha} w_k |w_{k,n}|^{p-2} w_{k,n} dX &\geq \int_{\mathbf{R}_+^{N+1}} t^{1-2\alpha} (|w_{k,n}|^{p/2})^2 dX, \\ \int_{\mathbf{R}^N} \lambda_\varepsilon \psi_k |\psi_{k,n}|^{p-2} \psi_{k,n} dx &\geq \int_{\mathbf{R}^N} \lambda_\varepsilon |\psi_{k,n}|^p dx, \\ \int_{\mathbf{R}^N} a_k \psi_k |\psi_{k,n}|^{p-2} \psi_{k,n} dx &= \left(\int_{[|\psi_k| < n]} + \int_{[|\psi_k| \geq n]} \right) a_k \psi_k |\psi_{k,n}|^{p-2} \psi_{k,n} dx \\ &\leq \int_{\mathbf{R}^N} |a_k| (|\psi_{k,n}|^{p/2})^2 dx + n^{p-1} k \int_{[|\psi_k| \geq n]} |\psi_k| dx \\ &\leq \varepsilon [|\psi_{k,n}|^{p/2}]_{H^\alpha}^2 + \lambda_\varepsilon \int_{\mathbf{R}^N} |\psi_{k,n}|^p dx + kn^{p-1} \int_{[|\psi_k| \geq n]} |\psi_k| dx. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} & \int_{\mathbf{R}_+^{N+1}} t^{1-2\alpha} \left\{ \frac{4}{p^2} (p-1) |\nabla (|w_{k,n}|^{p/2})|^2 + (|w_{k,n}|^{p/2})^2 \right\} dX \\ &\leq \kappa_\alpha \left[\varepsilon [|\psi_{k,n}|^{p/2}]_{H^\alpha}^2 + kn^{p-1} \int_{[|\psi_k| \geq n]} |\psi_k| dx + \lambda_\varepsilon \int_{\mathbf{R}^N} |u| |\psi_{k,n}|^{p-1} dx \right]. \end{aligned}$$

Since Proposition A.2 asserts

$$\kappa_\alpha \| |\psi_{k,n}|^{p/2} \|_{H^\alpha}^2 \leq \| |w_{k,n}|^{p/2} \|_{X^\alpha}^2, \quad n^{p-1} \leq |\psi_k|^{p-1} \quad \text{on } [|\psi_k| \geq n],$$

choosing

$$\varepsilon = \frac{1}{2} \frac{4(p-1)}{p^2},$$

we finally obtain

$$(70) \quad \| |w_{k,n}|^{p/2} \|_{X^\alpha}^2 \leq C_{p,\alpha} \left[k \int_{[|\psi_k| \geq n]} |\psi_k|^p dx + \int_{\mathbf{R}^N} |u| |\psi_{k,n}|^{p-1} dx \right].$$

Now let us consider the case where $u, \psi_k \in L^p(\mathbf{R}^N)$ and $\psi_k \rightarrow u$ strongly in $L^p(\mathbf{R}^N)$. By Hölder's inequality and the definition of $\psi_{k,n}$, we have

$$\int_{\mathbf{R}^N} |u| |\psi_{k,n}|^{p-1} dx \leq \|u\|_{L^p} \|\psi_k\|_{L^p}^{p-1}$$

and the right hand side in (70) is bounded as $n \rightarrow \infty$. Since $w_{k,n} \rightarrow w_k$ strongly in X^α , we observe that $|w_{k,n}|^{p/2} \rightharpoonup |w_k|^{p/2}$ weakly in X^α . Letting $n \rightarrow \infty$ in (70), one has

$$(71) \quad \| |w_k|^{p/2} \|_{X^\alpha}^2 \leq C_1 \|u\|_{L^p} \|\psi_k\|_{L^p}^{p-1}.$$

Since $\text{Tr}(|w_k|^{p/2}) = |\psi_k|^{p/2}$, Proposition A.2 gives

$$\kappa_\alpha \| |\psi_k|^{p/2} \|_{H^\alpha}^2 \leq \| |w_k|^{p/2} \|_{X^\alpha}^2 \leq C_1 \|u\|_{L^p} \|\psi_k\|_{L^p}^{p-1}.$$

Thus by Sobolev's inequality, we get

$$(72) \quad \| |\psi_k|^{p/2} \|_{L^{2^*_\alpha}^2}^2 \leq C_0 \| |\psi_k|^{p/2} \|_{H^\alpha}^2 \leq C \|u\|_{L^p} \| \psi_k \|_{L^p}^{p-1}.$$

Recalling $\psi_k \rightarrow u$ strongly in $L^p(\mathbf{R}^N)$ and letting $k \rightarrow \infty$ in (71), we observe that $(|w_k|^{p/2})$ is bounded in X^α , $|w_k|^{p/2} \rightharpoonup |w|^{p/2}$ weakly in X^α and

$$\| |w|^{p/2} \|_{X^\alpha}^2 \leq C_1 \|u\|_{L^p}^p$$

where $w = Eu \in X^\alpha$. Noting $\text{Tr}(|w|^{p/2}) = |u|^{p/2}$, we have

$$|u|^{p/2} \in H^\alpha(\mathbf{R}^N), \quad \kappa_\alpha \| |u|^{p/2} \|_{H^\alpha}^2 \leq \| |w|^{p/2} \|_{X^\alpha}^2 \leq C_1 \|u\|_{L^p}^p$$

and Step 3 holds.

Step 4: Conclusion

By Step 3 and (72), if $u, \psi_k \in L^p(\mathbf{R}^N)$ and $\psi_k \rightarrow u$ strongly in $L^p(\mathbf{R}^N)$, then

$$(73) \quad \| |u|^{p/2} \|_{L^{2^*_\alpha}^2}^2 \leq C \| |u|^{p/2} \|_{H^\alpha}^2 \leq C \kappa_\alpha^{-1} \| |w|^{p/2} \|_{X^\alpha}^2 \leq C \|u\|_{L^p}^p, \quad \| |\psi_k|^{p/2} \|_{L^{2^*_\alpha}^2}^2 \leq C \|u\|_{L^p} \| \psi_k \|_{L^p}^{p-1}.$$

Now we select $p = p_1 := 2^*_\alpha > 2$. From (69), the assumptions of Step 3 are satisfied and

$$\| |u|^{p_1/2} \|_{L^{2^*_\alpha}^2}^2 \leq C \| |u|^{p_1/2} \|_{H^\alpha}^2 \leq C \kappa_\alpha^{-1} \| |w|^{p_1/2} \|_{X^\alpha}^2 \leq C \|u\|_{L^{p_1}}^{p_1}, \quad \| |\psi_k|^{p_1/2} \|_{L^{2^*_\alpha}^2}^2 \leq C \|u\|_{L^{p_1}} \| \psi_k \|_{L^{p_1}}^{p_1-1}.$$

From this, one observes that the assumptions of Step 3 holds for any $2 \leq p < p_1 2^*_\alpha/2$. Hence, setting $p_2 := p_1 2^*_\alpha/2$, (73) holds for each $p < p_2$. Again, the assumptions of Step 3 hold for each $p < p_3 := p_2 2^*_\alpha/2$. Repeating this argument and noting $2^*_\alpha/2 > 1$, we observe that (73) holds for any $p < \infty$, which implies $u \in L^p(\mathbf{R}^N)$ for any $2 \leq p < \infty$. This completes the proof. \square

Acknowledgement. The author would like to thank Professor Tatsuki Kawakami, Professor Tohru Ozawa and Professor Kazunaga Tanaka for valuable comments and fruitful discussions on the topic of this paper.

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